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# Evolution and thermalization of dark matter axions in the condensed regime

Ken'ichi Saikawa[1,*] and Masahide Yamaguchi[2,†]
[1] *Institute for Cosmic Ray Research, The University of Tokyo, 5-1-5 Kashiwa-no-ha, Kashiwa City, Chiba 277-8582, Japan*
[2] *Department of Physics, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo 152-8551, Japan*
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We discuss the possibility that dark matter axions form a Bose-Einstein condensate (BEC) due to the gravitational self-interactions. The formation of BEC occurs in the condensed regime, where the transition rate between different momentum states is large compared to the energy exchanged in the transition. The time evolution of the quantum state occupation number of axions in the condensed regime is derived based on the in-in formalism. We recover the expression for the thermalization rate due to self-interaction of the axion field, which was obtained in the other literature. It is also found that the leading order contributions for interactions between axions and other species vanish, which implies that the axion BEC does not give any significant modifications on standard cosmological parameters.

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**Contents**



## I. INTRODUCTION

Identifying the origin of the dark matter of the universe is one of the priorities of modern high-energy physics and astronomy. So far, many particle physics models of the dark matter have been proposed (see e.g. [1] for reviews), and well motivated candidates are so called weakly interacting massive particles (WIMPs) and axions. Both of them possess suitable properties for the dark matter in that they are non-baryonic, cold, and collisionless. However, the nature of cosmological behavior is completely different between WIMPs and axions. WIMPs are produced from primordial soup of radiations, and their population is fixed when they decouple from the thermal plasma. We can interpret them

*Electronic address: saikawa@icrr.u-tokyo.ac.jp
†Electronic address: gucci@phys.titech.ac.jp

as a collection of classical particles, whose velocity dispersion is determined by the decoupling temperature. On the other hand, axions are produced non-thermally, having a small velocity dispersion compared with the temperature of the thermal plasma at the epoch when they are produced. Furthermore, they have huge occupation number in the phase space since they are bosons. Because of these properties, we interpret them as a classical field rather than individual particles. This classical field of axions coherently oscillating in the field space behaves as a cold matter component of the universe [2].

These curious properties of dark matter axions motivate the possibility that axions form a Bose-Einstein condensate (BEC) [3]. Indeed, the energy dispersion of axions at the production time $\delta\omega \sim \mathcal{O}(10^{-13})$eV is much smaller than the critical temperature for BEC, $T_c = (\pi^2 n/\zeta(3))^{1/3} \sim \mathcal{O}(100)$GeV, where $n$ is the number density of axions[1]. However, this argument relies on the following assumptions. First, the particles must be bosons. Second, their number must be conserved. Third, they should have huge phase space density. Finally, they must be in thermal equilibrium. The first three conditions are obviously satisfied for dark matter axions, but the final one, whether axions thermalize or not, is a non-trivial issue.

The cosmic thermalization of axions is extensively discussed in Ref. [4]. Here, the thermalization means that the system relaxes into a state with the highest entropy by exchanging energies and momenta between particle states. In order to investigate such a process, some statistical mechanical treatments are required. The usual analysis using the Boltzmann equation cannot be applied to this problem, since highly degenerate axions are essentially fields in the classical limit. Such a system of axion fields does not match the assumption of the Boltzmann analysis, where the system is considered to be a collection of point particles. This situation defines a peculiar regime of the many body system, called the condensed regime, which should be distinguished from the particle kinetic regime where the Boltzmann analysis can be applied. In the condensed regime, the energy transfer of the scattering process is small compared with the scattering rate, because the interaction occurs between highly degenerate states. Thermalization in the condensed regime is not well understood in comparison with that of the particle kinetic regime.

In Ref. [4], the thermalization process in the condensed regime is analyzed by describing the axion field as a set of quantum-mechanical oscillators (i.e. quantum operators) and deriving the evolution equation of each oscillator. However, in the formalism of [4] the thermalization rate is estimated by comparing the "order of magnitudes" of quantum operators, and the occurrence of the thermalization is confirmed only by computing the quantum-mechanical averages of the occupation number of each oscillator numerically, which is realized for toy models with small number of oscillators and particles. It is impossible to realize the actual system with huge number of axions using the numerical scheme described in [4].

In this paper, we revisit the issue of the axion thermalization. The purpose of this work is to develop a robust tool to describe the relaxation process of highly degenerate bosonic fields. Instead of using the approach of Ref. [4], we compute the expectation value (i.e. the quantum-mechanical average) of the occupation number of axions, and solve its time evolution, which informs us of the change of the distribution function of axions. The computation is executed by using the technology which was originally introduced by Schwinger et al. [5], called the "in-in" formalism. This formalism enables us to calculate the time evolution of the expectation value of quantum operator in the systematic way. Furthermore, it can treat a state to which particle-like interpretation is not applied, if we use an appropriate representation for a state at the initial time. In our case, a coherent state is used to describe a state in the condensed regime. Using this formalism, we estimate the thermalization rate as the inverse of the time scale in which the expectation value of the occupation number changes its value. Whole things can be described in the analytic way, and it is not necessary to use numerical simulations.

The discussion on thermalization of axions and formation of a BEC is not only the theoretical issue, but has a relevance to observations. There are several observational evidences indicating that the phase space structure of galactic halos is consistent with the "caustic ring model" [6]. In this model, the high density surfaces (caustics) in the phase-space distribution of the dark matter particles become ring-like configurations when dark matter particles fall into a galactic potential with net overall rotation. Recently, it was pointed out that this caustic ring model is predicted if dark matter axions form a BEC [7], which might be regarded as an evidence of the axion dark matter. However, it was also suggested that axion BEC might enter into thermal contact with photons, and modify some cosmological parameters from the standard values [4, 8]. In particular, if axions and photons reach thermal equilibrium, the photon temperature is cooled, which predicts a smaller value of the baryon to photon ratio at the big bang nucleosynthesis and larger value of the effective number of neutrino species $N_{\rm eff}$. The predicted value is $N_{\rm eff} = 6.77$, which is larger than the observed value $N_{\rm eff} \simeq$ 3-5 [9]. This result seems to be disapproval of axion BEC dark matter, but we find

[1] Here, the expression for the critical temperature $T_c = (\pi^2 n/\zeta(3))^{1/3}$ used in [3] is applied for relativistic particles, which might not be appropriate for cold axions because they are non-relativistic. Even though, the condition $\delta\omega \ll T_c$ is also satisfied if we use the expression for non-relativistic particles, $T_c = (2\pi/m)(n/\zeta(3/2))^{2/3} \sim \mathcal{O}(10^{22})$GeV where $m$ is the mass of the axion.

that such photon cooling effects are fictitious. As will be shown later, axions do not enter into thermal contact with photons, though they form a BEC.

The organization of this paper is as follows. In Sec. II, we introduce the method to evaluate the time evolution of the occupation number of axions. The expectation value of the quantum operator for the occupation number is computed by using the perturbative expansion in terms of the interaction Hamiltonian of the axion field. We perform the calculation of the leading order terms in perturbation theory, while the second order terms are evaluated in Appendix A. In Sec. III, implications of the results of our analysis on cosmology are discussed. We estimate the relaxation rate of self-interacting degenerate axions and recover the formula for the thermalization rate of axions, which was obtained in Ref. [4]. We will see that the thermalization rate exceeds the expansion rate when the temperature of the universe becomes sufficiently low, at which axion BEC is formed. Finally, summary and conclusions are given in Sec. IV.

## II. AXION FIELD DYNAMICS

In this section, we develop the formalism to compute the time evolution of quantum occupation number of the axion field. Our interest is to calculate the expectation value of a quantum operator $\hat{\mathcal{N}}_{\mathbf{p}}(t)$ at a given time $t$, which represents the number of axions occupying a quantum state labeled by three-momentum $\mathbf{p}$. Such a problem can be dealt by using the "in-in" formalism (or Schwinger-Keldysh formalism) [5]. In cosmology, this formalism was applied to calculate quantum contributions to cosmological correlations [10, 11]. Following such a formalism closely, in this work, we calculate the time evolution of the occupation number in a systematic way by use of the perturbative expansion.

The outline of this section is as follows. In Sec. II A, we review the formalism described in [10, 11] and give the formula to calculate the expectation value of the quantum operator. In Sec. II B, the mode expansion of the field operator is taken and the quantum occupation number is defined. In Sec. II C, we discuss how to represent the coherently oscillating axion fields as quantum states. Using the ingredients obtained in Secs. II A, II B, and II C, we compute the time evolution of the occupation number due to the self-interaction of the axion field in Sec. II D. Finally, interactions with other particles such as baryons and photons are discussed in Sec. II E.

### A. The in-in formalism

Let us consider the theory with a real scalar field (the axion field) $\phi(\mathbf{x},t)$ in the Minkowski background. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_I, \tag{1}$$

where $m$ is the mass of the axion and $\mathcal{L}_I$ is the interaction term which we specify later. The Hamiltonian of the system is given by

$$H[\phi(t),\pi(t)] = \int d^3x(\pi\dot{\phi} - \mathcal{L}) = H_0[\phi(t),\pi(t)] + H_I[\phi(t),\pi(t)], \tag{2}$$

where $\pi(\mathbf{x},t) = \dot{\phi}(\mathbf{x},t)$ is the canonical conjugate. Here, we decompose $H$ into its free and interaction terms,

$$H_0[\phi(t),\pi(t)] = \int d^3x\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2\right], \tag{3}$$

$$H_I[\phi(t),\pi(t)] = -\int d^3x\mathcal{L}_I. \tag{4}$$

The quantum operators satisfy the canonical commutation relations,

$$\begin{aligned}&[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\ &[\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0.\end{aligned} \tag{5}$$

Their time evolution is given by the Heisenberg equations,

$$\begin{aligned}\dot{\phi}(\mathbf{x},t) &= i\left[H[\phi(t),\pi(t)],\phi(\mathbf{x},t)\right],\\ \dot{\pi}(\mathbf{x},t) &= i\left[H[\phi(t),\pi(t)],\pi(\mathbf{x},t)\right].\end{aligned} \tag{6}$$

These equations can be solved formally,

$$\begin{aligned} \phi(t) &= U^{-1}(t,t_0)\phi(t_0)U(t,t_0), \\ \pi(t) &= U^{-1}(t,t_0)\pi(t_0)U(t,t_0), \end{aligned} \tag{7}$$

for some fixed time $t_0$, where $U(t,t_0)$ is given by

$$\frac{d}{dt}U(t,t_0) = -iH[\phi(t),\pi(t)]U(t,t_0) \quad \text{and} \quad U(t_0,t_0) = 1. \tag{8}$$

Now we move on to the interaction picture. Let us define interaction picture fields $\phi^I$ and $\pi^I$ such that

$$\begin{aligned} \dot{\phi}^I(\mathbf{x},t) &= i\left[H_0[\phi^I(t),\pi^I(t)],\phi^I(\mathbf{x},t)\right], \\ \dot{\pi}^I(\mathbf{x},t) &= i\left[H_0[\phi^I(t),\pi^I(t)],\pi^I(\mathbf{x},t)\right], \end{aligned} \tag{9}$$

with $\phi^I(t_0) = \phi(t_0)$ and $\pi^I(t_0) = \pi(t_0)$. Solutions of these equations are given by

$$\begin{aligned} \phi^I(t) &= U_0^{-1}(t,t_0)\phi(t_0)U_0(t,t_0), \\ \pi^I(t) &= U_0^{-1}(t,t_0)\pi(t_0)U_0(t,t_0), \end{aligned} \tag{10}$$

where $U_0(t,t_0)$ satisfies the following equation,

$$\frac{d}{dt}U_0(t,t_0) = -iH_0[\phi^I(t),\pi^I(t)]U(t,t_0) \quad \text{and} \quad U_0(t_0,t_0) = 1. \tag{11}$$

By noting that

$$\begin{aligned} H_0[\phi^I(t),\pi^I(t)] &= H_0[\phi(t_0),\pi(t_0)], \\ H[\phi(t),\pi(t)] &= H[\phi(t_0),\pi(t_0)], \end{aligned} \tag{12}$$

Eqs. (8) and (11) lead to

$$\frac{d}{dt}F(t,t_0) = -iH_I(t)F(t,t_0) \quad \text{and} \quad F(t_0,t_0) = 1, \tag{13}$$

where

$$F(t,t_0) \equiv U_0^{-1}(t,t_0)U(t,t_0), \tag{14}$$

$$H_I(t) \equiv U_0^{-1}(t,t_0)H_I[\phi(t_0),\pi(t_0)]U_0(t,t_0) = H_I[\phi^I(t),\pi^I(t)]. \tag{15}$$

The solution of Eq. (13) is given by

$$F(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t)dt\right), \tag{16}$$

and also

$$F^{-1}(t,t_0) = \bar{T}\exp\left(i\int_{t_0}^{t} H_I(t)dt\right), \tag{17}$$

where $T$ ($\bar{T}$) represents (anti-)time ordering.

A quantum operator $\mathcal{O}[\phi(t),\pi(t)]$ constructed from $\phi$ and $\pi$ can be written as

$$\begin{aligned} \mathcal{O}(t) &= F^{-1}(t,t_0)\mathcal{O}^I(t)F(t,t_0) \\ &= \left[\bar{T}\exp\left(i\int_{t_0}^{t} H_I(t)dt\right)\right]\mathcal{O}^I(t)\left[T\exp\left(-i\int_{t_0}^{t} H_I(t)dt\right)\right], \end{aligned} \tag{18}$$

where $\mathcal{O}^I(t) \equiv \mathcal{O}[\phi^I(t),\pi^I(t)]$. Using Eq. (18), we can compute the expectation value of the operator $\langle\mathcal{O}(t)\rangle = \langle\Psi|\mathcal{O}(t)|\Psi\rangle$ at a given time $t$ for an "in" state $|\Psi\rangle$ specified at the time $t_0$. It is convenient to note that

$$\langle\mathcal{O}(t)\rangle = \sum_{N=0}^{\infty} i^N \int_{t_0}^{t} dt_N \int_{t_0}^{t_N} dt_{N-1} \ldots \int_{t_0}^{t_2} dt_1 \langle[H_I(t_1),[H_I(t_2),\ldots[H_I(t_N),\mathcal{O}^I(t)]\ldots]]\rangle, \tag{19}$$

which can be derived by mathematical induction.

### B. Mode expansion and occupation number

Since the calculation in Eq. (19) is performed in terms of the interaction picture fields $\phi^I$ and $\pi^I$, it is convenient to write down relevant quantities in the interaction picture. From Eqs. (5) and (10), the interaction picture fields also satisfy the commutation relations,

$$[\phi^I(\mathbf{x},t),\pi^I(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$[\phi^I(\mathbf{x},t),\phi^I(\mathbf{y},t)] = [\pi^I(\mathbf{x},t),\pi^I(\mathbf{y},t)] = 0. \tag{20}$$

Eqs. (9) imply that $\phi^I$ and $\pi^I$ are solutions of free field equations of motion, giving their mode expansions,

$$\phi^I(\mathbf{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[e^{ip\cdot x}a^I_{\mathbf{p}} + e^{-ip\cdot x}a^{I\dagger}_{\mathbf{p}}\right], \tag{21}$$

$$\pi^I(\mathbf{x},t) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_p}{2}}\left[e^{ip\cdot x}a^I_{\mathbf{p}} - e^{-ip\cdot x}a^{I\dagger}_{\mathbf{p}}\right], \tag{22}$$

where $E_p = \sqrt{m^2+|\mathbf{p}|^2}$, $x^0 = t$, and $p^0 = E_p$. Then, the commutation relations (20) are equivalent to

$$[a^I_{\mathbf{p}}, a^{I\dagger}_{\mathbf{p}'}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}'), \quad \text{and} \quad [a^I_{\mathbf{p}}, a^I_{\mathbf{p}'}] = [a^{I\dagger}_{\mathbf{p}}, a^{I\dagger}_{\mathbf{p}'}] = 0. \tag{23}$$

The creation and annihilation operators diagonalize the free Hamiltonian of interaction picture fields,

$$H_0[\phi^I(t),\pi^I(t)] = \int d^3x\left[\frac{1}{2}(\pi^I)^2 + \frac{1}{2}(\nabla\phi^I)^2 + \frac{1}{2}m^2(\phi^I)^2\right] = \int \frac{d^3p}{(2\pi)^3}E_p\left(a^{I\dagger}_{\mathbf{p}}a^I_{\mathbf{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right). \tag{24}$$

Here, let us define the operator whose eigenvalue gives occupation number of a momentum state $\mathbf{p}$,

$$\hat{\mathcal{N}}_{\mathbf{p}} \equiv \frac{d^3p}{(2\pi)^3}a^{I\dagger}_{\mathbf{p}}a^I_{\mathbf{p}}. \tag{25}$$

On the other hand, its eigenstate can be obtained by applying the ladder operator $a^{I\dagger}_{\mathbf{p}}$ on the vacuum state defined by

$$a^I_{\mathbf{p}}|0\rangle_I = 0. \tag{26}$$

An operator similar to (25) in the Heisenberg picture can also be constructed. Since Heisenberg picture and interaction picture operators are related,

$$\phi(\mathbf{x},t) = F^{-1}(t,t_0)\phi^I(\mathbf{x},t)F(t,t_0) \quad \text{and} \quad \pi(\mathbf{x},t) = F^{-1}(t,t_0)\pi^I(\mathbf{x},t)F(t,t_0), \tag{27}$$

the following time-dependent operators are useful,

$$a_{\mathbf{p}}(t) = F^{-1}(t,t_0)a^I_{\mathbf{p}}F(t,t_0) \quad \text{and} \quad a^\dagger_{\mathbf{p}}(t) = F^{-1}(t,t_0)a^{I\dagger}_{\mathbf{p}}F(t,t_0). \tag{28}$$

From Eqs. (23) and (28), it is manifest that $a_{\mathbf{p}}(t)$ and $a^\dagger_{\mathbf{p}}(t)$ also satisfy the canonical commutation relations, and diagonalize the free Hamiltonian of Heisenberg picture fields $H_0[\phi(t),\pi(t)]$. Hence we recognize that the operator

$$\hat{\mathcal{N}}_{\mathbf{p}}(t) \equiv \frac{d^3p}{(2\pi)^3}a^\dagger_{\mathbf{p}}(t)a_{\mathbf{p}}(t) \tag{29}$$

describes the time evolution of the occupation number, and we substitute it into $\mathcal{O}(t)$ in the left hand side of Eq. (19). However, in the actual calculation, the operator (25) is used as $\mathcal{O}^I$ in the right hand side of Eq. (19).

Note that neither $a^I_{\mathbf{p}}$ nor $a_{\mathbf{p}}(t)$ diagonalizes the full Hamiltonian $H = H_0 + H_I$, and the state $|0\rangle_I$ given by Eq. (26) is not the ground state of the full Hamiltonian, which we shall denote $|0\rangle_H$. On the other hand, the "in" state, which is used to calculate the expectation value in Eq. (19), is defined as an eigenstate of $H$, not $H_0$. Such a state can be constructed by applying the operator $a^{\mathrm{in}\dagger}_{\mathbf{p}}$, which creates one particle state from $|0\rangle_H$. It is assumed that this "in" state approaches a free particle state constructed by $a^\dagger(t)$ in the limit $t_0 - t \to -\infty$, up to a factor representing the renormalization of the wave function. Such a factor can be absorbed into the physical mass of the field $\phi$, which differs from the bare mass $m$ appealing in $H_0$ [12].

Aside from the renormalization factor, in the limit $t_0 - t \to -\infty$, the right hand side of Eq. (19) can be reduced into the expectation value in the "vacuum" $|0\rangle_I$ multiplied by a factor arising from the overlap between states $|0\rangle_I$ and $|0\rangle_H$. This overlapping factor drops out when we divide the expectation value by $1 =_H \langle 0|0\rangle_H$. This procedure is justified by taking the limit $t_0 - t \to -\infty(1 - i\epsilon)$ in a slightly imaginary direction, where $\epsilon$ is a positive infinitesimal. Note that, in this case, the quantity appearing in the denominator is equal to unity because of ${}_I\langle 0|F^{-1}(t,t_0)F(t,t_0)|0\rangle_I = 1$. This fact implies that all vacuum fluctuations automatically vanish in the in-in formalism [10].

Having removed ambiguities via the procedure described above, we can simply calculate the right hand side of Eq. (19) with the state constructed by applying operators $a_{\mathbf{p}}^{I\dagger}$ on the "vacuum" state $|0\rangle_I$ in the interaction picture. This initial state will be specified in the next subsection.

In the following, the subscript "$I$" is omitted for simplicity. It is convenient to consider a finite spatial box with volume $V = L^3$ so that the label of each mode becomes discrete, $\mathbf{p_n} = (2\pi/L)\mathbf{n}$, and $\mathbf{n} = (n_x, n_y, n_z)$, where $n_x$, $n_y$ and $n_z$ are integers. Then we just take the following replacements,

$$(2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}') \to V\delta_{n_x,n'_x}\delta_{n_y,n'_y}\delta_{n_z,n'_z},$$

$$\int \frac{d^3p}{(2\pi)^3} \to \frac{1}{V}\sum_{n_x,n_y,n_z},$$

and also

$$[a_i, a_j^\dagger] = V\delta_{i,j}, \quad \text{and} \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0. \tag{30}$$

Here, the italic indices $i$, $j$ should be understood as abbreviated notation for three dimensional vectors with discrete components. The number operator defined in Eq. (25) is rewritten as

$$\hat{\mathcal{N}}_n \equiv \frac{a_n^\dagger a_n}{V}, \tag{31}$$

where the factor $1/V$ appears due to the factor $V$ in Eq. (30).

### C. Coherent oscillation as a quantum state

Now let us specify the state at the initial time. Since we are interested in the evolution of coherently oscillating axions, the initial time is set to be the epoch of the QCD phase transition, at which the mass $m$ becomes greater than the Hubble parameter $H$ and the classical axion field begins to oscillate around the minimum of the potential. These axions are produced due to the misalignment mechanism [2], and are called the zero mode, since a huge number of axions homogeneously oscillate over a large distance.

In addition to this zero mode, however, there are additional contributions to the axion abundance. One contribution is produced by the thermal bath in the early universe, and its abundance is fixed at the decoupling temperature [13]. Another contribution comes from the decay of topological defects, such as global strings and domain walls [14, 15]. Both of them have definite momenta, and we call them the non-zero modes in contrast to the zero mode. If inflation occurred before the Peccei-Quinn (PQ) phase transition, those produced by topological defects can be a dominant component of dark matter. On the other hand, if inflation occurred after the PQ phase transition, their population is negligible. See [16–18] for recent developments about this issue.

Each of the zero mode and the non-zero modes corresponds to a definite quantum state. In particular, it is possible to construct a state with a definite momentum $\mathbf{p_k}$ occupied by $\mathcal{N}_k$ axions as a *number state*

$$|\mathcal{N}_k\rangle = \frac{1}{\sqrt{\mathcal{N}_k! V^{\mathcal{N}_k}}}(a_k^\dagger)^{\mathcal{N}_k}|0\rangle_I, \tag{32}$$

where $|0\rangle_I$ is the vacuum defined by Eq. (26). This is an eigenstate of the number operator (31), and we can construct complete orthonormal basis by using a series of the number states. Here, it should be noted that the non-zero modes correspond to the number states. In the classical limit, these states can be interpreted as classical point particles with definite energies and momenta.

On the other hand, the zero mode has different properties compared with non-zero modes. It has a huge occupation number as large as $\mathcal{N} \sim 10^{61}$. In the classical limit, this state should be interpreted as a classical field, rather than point particles [4]. Such a highly degenerate Bose gas of axions might be described as a *coherent state* [19]. Mathematically, a coherent state can be represented by using the basis of number states [20]

$$|\alpha_i\rangle = e^{-\frac{1}{2}|\alpha_i|^2}\sum_{n=0}^{\infty}\frac{\alpha_i^n}{n!\sqrt{V^n}}(a_i^\dagger)^n|0\rangle_I, \tag{33}$$

where $\alpha_i$ is a complex number and the numerical factor is chosen so that it is normalized $\langle\alpha_i|\alpha_i\rangle = 1$. The coherent state is characterized as an eigenvector of the annihilation operator such that

$$a_i|\alpha_j\rangle = V^{1/2}\alpha_j\delta_{ij}|\alpha_j\rangle. \tag{34}$$

Hereafter we assume that a huge number of particles occupy a small number $K$ of states around the ground state and that they are described as coherent states. There also exist non-zero modes, which occupy states with higher momenta and are described as number states. The collection of such states can be expressed as

$$|\{\mathcal{N}\},\{\alpha\}\rangle = \prod_{k>K}\frac{1}{\sqrt{\mathcal{N}_k!V^{\mathcal{N}_k}}}(a_k^\dagger)^{\mathcal{N}_k}|\{\alpha\}\rangle, \tag{35}$$

$$|\{\alpha\}\rangle = \prod_{i\le K} e^{-\frac{1}{2}|\alpha_i|^2}\sum_{n=0}^{\infty}\frac{\alpha_i^n}{n!\sqrt{V^n}}(a_i^\dagger)^n|0\rangle_I. \tag{36}$$

Note that $i \le K$ is the abbreviated notation representing the sum over lowest $K$ modes (i.e. actual states are labeled by three momenta, and we must distinguish them by spatial directions of momenta as well as their absolute value). Let us call the modes with $i > K$ the particle-like modes and the modes with $i \le K$ the condensed modes. It would be convenient to note the following relations:

$$\left[a_i,(a_j^\dagger)^{\mathcal{N}_j}\right] = \mathcal{N}_j V\delta_{ij}(a_j^\dagger)^{\mathcal{N}_j-1}, \tag{37}$$

$$\left[(a_i)^{\mathcal{N}_i},a_i^\dagger\right] = \mathcal{N}_j V\delta_{ij}(a_j)^{\mathcal{N}_j-1}, \tag{38}$$

$$a_k|\{\mathcal{N}\},\{\alpha\}\rangle = \begin{cases}\sqrt{\mathcal{N}_k V}|\{\mathcal{N}\}^k,\{\alpha\}\rangle & \text{if} \quad k > K\\ \alpha_k\sqrt{V}|\{\mathcal{N}\},\{\alpha\}\rangle & \text{if} \quad k \le K\end{cases}, \tag{39}$$

where $|\{\mathcal{N}\}^k,\{\alpha\}\rangle$ is the state obtained by replacing the factor $(a_k^\dagger)^{\mathcal{N}_k}/\sqrt{\mathcal{N}_k!V^{\mathcal{N}_k}}$ with $(a_k^\dagger)^{\mathcal{N}_k-1}/\sqrt{(\mathcal{N}_k-1)!V^{\mathcal{N}_k-1}}$ in Eq. (35).

It is important to assume that there are plural condensed modes ($K > 1$). Our interest is to know how these condensed modes reach thermal equilibrium by exchanging their momenta. In this sense, the "zero mode" is not exactly a single mode with zero momentum, but the collection of $K$ plural modes near the ground state. We summarize the contents of the initial state in table I.

TABLE I: Classification of axions with their origins and quantum state representations.

| | production mechanism | quantum state |
|---|---|---|
| zero mode | misalignment mechanism | coherent states (condensed modes) |
| non-zero mode (topological defects) | decay of defects | number states (particle-like modes) |
| non-zero mode (thermal axions) | thermal decoupling | number states (particle-like modes) |

Let us take the expectation value of $\phi$ given by Eq. (21) at the initial time $t_0$ for the state $|\{\mathcal{N}\},\{\alpha\}\rangle$,

$$\begin{aligned}\phi_0 &\equiv \langle\{\mathcal{N}\},\{\alpha\}|\phi(\mathbf{x},t_0)|\{\mathcal{N}\},\{\alpha\}\rangle\\ &= \sum_{n\le K}\frac{1}{\sqrt{2E_nV}}(e^{-iE_nt_0+i\mathbf{p_n}\cdot\mathbf{x}}\alpha_n + e^{iE_nt_0-i\mathbf{p_n}\cdot\mathbf{x}}\alpha_n^*).\end{aligned} \tag{40}$$

Since the wavelength of the condensed modes is comparable or greater than the QCD horizon, $|\mathbf{p_n}| \lesssim H(t_0) \sim t_0^{-1}$, $\mathbf{p_n}\cdot\mathbf{x} \ll 1$ and hence the factor $e^{\pm i\mathbf{p_n}\cdot\mathbf{x}}$ is negligible. This approximation remains valid as long as we consider the dynamics inside the horizon. We also approximate $E_n = \sqrt{m^2+p_n^2} \simeq m$ since the coherent oscillation begins when $|\mathbf{p_n}| \lesssim H(t_0) < m$ is satisfied. Then, the expectation value, $\phi_0$, is given by

$$\begin{aligned}\phi_0 &\simeq \sum_{n\le K}\frac{1}{\sqrt{2mV}}(e^{-imt_0}\alpha_n + e^{imt_0}\alpha_n^*)\\ &= \sum_{n\le K}\sqrt{\frac{2}{mV}}|\alpha_n|\cos(mt_0-\beta_n),\end{aligned} \tag{41}$$

with

$$\alpha_n = |\alpha_n| e^{i\beta_n}. \tag{42}$$

If the condensed modes are decoupled with each other, the expectation value of the field oscillates like $\langle\phi\rangle \propto \cos(mt - \beta_n)$. Each mode oscillates independently with different amplitude $|\alpha_n|$ and the total amplitude is given by the superposition of $K$ oscillating modes.

Next, let us take the mean square deviation of the field amplitude for a single coherent state given in Eq. (33),

$$\begin{aligned}\Delta\phi &= \sqrt{\langle\alpha_i|\phi^2|\alpha_i\rangle - \langle\alpha_i|\phi|\alpha_i\rangle^2} \\ &= \sqrt{\frac{1}{V}\sum_n \frac{1}{2E_n}} \xrightarrow{V\to\infty} \sqrt{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}}.\end{aligned} \tag{43}$$

Since this result does not depend on $\alpha_i$, it holds for the state with $\alpha_i = 0$ (the vacuum state), which implies that the deviation given in Eq. (43) is nothing but the vacuum fluctuation. Therefore, the coherent state has the same trajectory with the classical field and the same fluctuation with the vacuum.

The expectation value of the momentum conjugate (22) leads to

$$\begin{aligned}\dot{\phi}_0 &\equiv \langle\{\mathcal{N}\},\{\alpha\}|\pi(\mathbf{x},t_0)|\{\mathcal{N}\},\{\alpha\}\rangle \\ &= \sum_{n\le K}\sqrt{\frac{2m}{V}}|\alpha_n|\sin(\beta_n - mt_0).\end{aligned} \tag{44}$$

Let us assume that the initial velocity $\langle\dot{\phi}\rangle$ of every mode vanishes, $\beta_n = mt_0$. In this case, we obtain

$$\phi_0 \simeq \sum_{n\le K}\sqrt{\frac{2}{mV}}|\alpha_n| = \sum_{n\le K}\theta_n^{\rm ini}F_a, \tag{45}$$

where $F_a$ is the axion decay constant and

$$\theta_n^{\rm ini} \equiv \sqrt{\frac{2}{mV}}\frac{|\alpha_n|}{F_a} \tag{46}$$

is the initial misalignment angle for a mode $n$.

The total number of particles at the initial time is given by

$$N = \sum_n \langle\{\mathcal{N}\},\{\alpha\}|\hat{\mathcal{N}}_n|\{\mathcal{N}\},\{\alpha\}\rangle, \tag{47}$$

where $\hat{\mathcal{N}}_n$ is the number operator given in Eq. (31). Dividing it by a volume $V$ yields the number density of axions

$$n_{\rm tot} = \frac{N}{V} = \frac{1}{V^2}\sum_n\langle\{\mathcal{N}\},\{\alpha\}|a_n^\dagger a_n|\{\mathcal{N}\},\{\alpha\}\rangle = n_p + n_c, \tag{48}$$

where

$$n_p \equiv \frac{1}{V}\sum_{n>K}\mathcal{N}_n \tag{49}$$

is the number density of particle-like modes, and

$$n_c \equiv \sum_{n\le K} n_{c,n} = \frac{1}{2}mF_a^2(\theta^{\rm ini})^2, \quad n_{c,n} \equiv \frac{1}{V}|\alpha_n|^2 \tag{50}$$

are the number densities of condensed modes. Here $(\theta^{\rm ini})^2$ is the square of the total misalignment angle

$$(\theta^{\rm ini})^2 \equiv \sum_{n\le K}(\theta_n^{\rm ini})^2. \tag{51}$$

In the continuous limit $V \to \infty$, Eq. (48) can be rewritten as

$$n_{\rm tot} = \int \frac{d^3p}{(2\pi)^3} f(\mathbf{p}), \tag{52}$$

where $f(\mathbf{p})$ is the total phase space distribution function of axions,

$$f(\mathbf{p}) = \mathcal{N}_{\mathbf{p}} + \sum_{n \leq K} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p_n}) n_{c,n}. \tag{53}$$

### D. Time evolution of quantum occupation number

In the following, we compute the time evolution of the expectation value of the quantum number operator (19). By using the in-in formalism, the time evolution of the occupation number is given by

$$\langle \hat{\mathcal{N}}_p(t) \rangle = \langle \hat{\mathcal{N}}_p \rangle + i \int_{t_0}^{t} dt_1 \langle [H_I(t_1), \hat{\mathcal{N}}_p] \rangle - \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \langle [H_I(t_1), [H_I(t_2), \hat{\mathcal{N}}_p]] \rangle + (\text{higher order in } H_I), \tag{54}$$

where $\langle \dots \rangle$ represents the expectation value for the state given by Eq. (35). We consider the following form of the interaction [4],

$$H_I(t) = \frac{1}{V^4} \sum_{ijkl} \frac{1}{4} \Lambda^{ij}_{kl} e^{-i\Omega^{ij}_{kl} t} a^\dagger_k a^\dagger_l a_i a_j, \tag{55}$$

where $\Omega^{ij}_{kl} \equiv E_i + E_j - E_k - E_l$ and $\Lambda^{ij}_{kl}$ satisfies $\Lambda^{ij}_{kl} = \Lambda^{ji}_{kl} = \Lambda^{ij}_{lk} = \Lambda^{kl*}_{ij}$. This can be obtained from $\lambda\phi^4/4!$ type interaction in the effective Lagrangian of the axion field with $\lambda \simeq 0.35 m^2/F_a^2$, and the coefficient $\Lambda^{ij}_{kl}$ becomes

$$\Lambda_s{}^{ij}{}_{kl} = -\frac{\lambda}{4\sqrt{E_i E_j E_k E_l}} V \delta_{i+j,k+l}. \tag{56}$$

Here we dropped the processes which violate axion number such as $a^\dagger a^\dagger a^\dagger a$, since the rate of such a process is too small and irrelevant for the case of interest [4]. In addition to this self-coupling, axions also interact due to their gravitational potential. In the Newtonian limit, the interaction Hamiltonian of the gravitational coupling is given by

$$H_{I,g}[\phi(t), \pi(t)] = -\frac{G}{2} \int d^3x d^3x' \frac{\rho(\mathbf{x}, t)\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}, \tag{57}$$

where $G$ is the Newton's constant and $\rho(\mathbf{x}, t) = (\pi^2(\mathbf{x}, t) + m^2\phi^2(\mathbf{x}, t))/2$ is the energy density of axions. This leads to the term (55) with the coefficient

$$\Lambda_g{}^{ij}{}_{kl} = -4\pi G m^2 \left( \frac{1}{|\mathbf{p}_k - \mathbf{p}_i|^2} + \frac{1}{|\mathbf{p}_k - \mathbf{p}_j|^2} \right) V \delta_{i+j,k+l}, \tag{58}$$

where we used the approximation $E_i \approx m$.

Let us evaluate the term of first order in $H_I$. Using the following relation,

$$[a^\dagger_k a^\dagger_l a_i a_j, a^\dagger_p a_p] = V\delta_{ip} a^\dagger_k a^\dagger_l a_j a_p + V\delta_{jp} a^\dagger_k a^\dagger_l a_i a_p - V\delta_{kp} a^\dagger_p a^\dagger_l a_i a_j - V\delta_{lp} a^\dagger_p a^\dagger_k a_i a_j, \tag{59}$$

the commutation relation becomes

$$[H_I(t), \hat{\mathcal{N}}_p] = \frac{1}{2V^4} \sum_{jkl} \left[ \Lambda^{pj}_{kl} e^{-i\Omega^{pj}_{kl} t} a^\dagger_k a^\dagger_l a_j a_p - \text{H.c.} \right]. \tag{60}$$

The following relations coming from Eq. (39) are useful to take the expectation value of the above term,

$$a_{k'} a_k |\{\mathcal{N}\}, \{\alpha\}\rangle = \begin{cases} \sqrt{\mathcal{N}_k(\mathcal{N}_k - 1)} V |\{\mathcal{N}\}^{2k}, \{\alpha\}\rangle & \text{if} \;\; k = k' > K \\ \sqrt{\mathcal{N}_k \mathcal{N}_{k'}} V |\{\mathcal{N}\}^{k,k'}, \{\alpha\}\rangle & \text{if} \;\; k \neq k', \;\; k > K, \text{ and } k' > K \\ \sqrt{\mathcal{N}_k} \alpha_{k'} V |\{\mathcal{N}\}^{k}, \{\alpha\}\rangle & \text{if} \;\; k > K \text{ and } k' \leq K \\ \alpha_k \alpha_{k'} V |\{\mathcal{N}\}, \{\alpha\}\rangle & \text{if} \;\; k \leq K, \text{ and } k' \leq K \end{cases}, \tag{61}$$

where the state $|\{\mathcal{N}\}^{2k},\{\alpha\}\rangle$ contains a factor $(a_k^\dagger)^{\mathcal{N}_k-2}/\sqrt{(\mathcal{N}_k-2)!V^{\mathcal{N}_k-2}}$ for mode $k$, and the state $|\{\mathcal{N}\}^{k,k'},\{\alpha\}\rangle$ contains a factor $(a_k^\dagger)^{\mathcal{N}_k-1}(a_{k'}^\dagger)^{\mathcal{N}_{k'}-1}/\sqrt{(\mathcal{N}_k-1)!V^{\mathcal{N}_k-1}(\mathcal{N}_{k'}-1)!V^{\mathcal{N}_{k'}-1}}$ for modes $k$ and $k'$. By separating the summation over indices $jkl$ into the contribution of particle-like modes $>K$ and that of condensed modes $\leq K$, and using Eq. (61), we can compute the expectation value of Eq. (60). For $p\leq K$, after some algebra, we obtain

$$\begin{aligned}\langle[H_I(t),\hat{\mathcal{N}_p}]\rangle =&\frac{1}{V^2}\sum_{j>K}\sum_{k\leq K}\left[\Lambda^{pj}_{kj}e^{-i(E_p-E_k)t}\mathcal{N}_j\alpha_p\alpha_k^*-\text{c.c.}\right]\\ &+\frac{1}{2V^2}\sum_{j\leq K}\sum_{k\leq K}\sum_{l\leq K}\left[\Lambda^{pj}_{kl}e^{-i\Omega^{pj}_{kl}t}\alpha_k^*\alpha_l^*\alpha_j\alpha_p-\text{c.c.}\right]\quad\text{for}\quad p\leq K.\end{aligned}\tag{62}$$

The first term in the right hand side of Eq. (62) vanishes since $\Lambda^{pj}_{kj}$ contains the conservation law of three momenta $\delta_{p+j,k+j}$. On the other hand, for $p>K$, this term exactly vanishes

$$\langle[H_I(t),\hat{\mathcal{N}_p}]\rangle=0\quad\text{for}\quad p>K.\tag{63}$$

Finally, taking the time integration yields the contribution at the first order perturbation,

$$\begin{aligned}i\int_{t_0}^t dt_1\langle[H_I(t),\hat{\mathcal{N}_p}]\rangle &=\frac{1}{2V^2}\sum_{j\leq K}\sum_{k\leq K}\sum_{l\leq K}\left[\Lambda^{pj}_{kl}\frac{e^{-i\Omega^{pj}_{kl}t}(e^{i\Omega^{pj}_{kl}(t-t_0)}-1)}{\Omega^{pj}_{kl}}\alpha_k^*\alpha_l^*\alpha_j\alpha_p+\text{c.c.}\right]\\ &\simeq-\frac{1}{2V^2}\sum_{j\leq K}\sum_{k\leq K}\sum_{l\leq K}\left[\Lambda^{pj}_{kl}\frac{e^{-i\Omega^{pj}_{kl}t}}{\Omega^{pj}_{kl}}\alpha_k^*\alpha_l^*\alpha_j\alpha_p+\text{c.c.}\right]\quad\text{for}\quad p\leq K,\end{aligned}\tag{64}$$

and

$$i\int_{t_0}^t dt_1\langle[H_I(t),\hat{\mathcal{N}_p}]\rangle=0\quad\text{for}\quad p>K,\tag{65}$$

where we have dropped the rapidly oscillating term $e^{i\Omega^{pj}_{kl}(t-t_0)}$ as $t-t_0\to\infty$ in the second line of Eq. (64). Note that, if there is no scattering ($p=k$ or $p=l$), the first line of Eq. (64) also vanishes.

As was conjectured in [4], there are two distinct regimes for the interaction process. One is the particle kinetic regime characterized by the condition $\Gamma\ll\delta\omega$, where $\Gamma$ is the evolution rate of the system and $\delta\omega$ is the typical energy exchanged in the interaction. In this regime we expect that $\Omega^{pj}_{kl}t\gg1$ and the factor $\exp(-i\Omega^{pj}_{kl}t)$ in Eq. (64) cancels out when the time average is taken. Hence the first order term (64) becomes irrelevant, which requires us to evaluate second order terms in the expansion (54) to follow the time evolution of the occupation number. The explicit calculation for second order terms is given in Appendix A.

The opposite regime characterized by the condition $\Gamma\gg\delta\omega$ is called the condensed regime. Since $\Omega^{pj}_{kl}t\ll1$ is satisfied, we can safely set

$$e^{-i\Omega^{pj}_{kl}t}\simeq1\tag{66}$$

in Eq. (64), and hence the first order term becomes relevant for the estimation of the evolution rate. In this regime, considerably small momenta $\delta\omega$ are exchanged between $K$ highly occupied states. It makes sense even though $\Omega^{pj}_{kl}t\ll1$, since the transition occurs as $\mathcal{N}\Omega^{pj}_{kl}t\gg1$ for huge number of particles, $\mathcal{N}$. The expression (64) will be used in estimating the thermalization rate of axions in Sec. III.

### E. Interaction with other species

So far we have considered only the self-interaction of the axion field. The highly degenerate axions may also couple with other species, such as baryons, relativistic axions, and photons due to the gravitational interactions, though the coupling with relativistic axions has already been included in the formalism described in the previous subsection. In Ref. [4], it is claimed that axions have thermal contact with other species after they form a BEC. However, as shown in this subsection, there are no such effects at least at the first order in perturbation theory.

In general, the interaction Hamiltonian with other species $b$ can be written as

$$H_{I,b}(t)=\frac{1}{V^4}\sum_{ijkl}\frac{1}{4}\Lambda_b{}^{ij}{}_{kl}e^{-i\Omega^{ij}_{kl}t}a_k^\dagger b_l^\dagger a_ib_j,\tag{67}$$

where ${\Lambda_b}^{ij}{}_{kl}$ is a constant which contains the conservation law of three momenta, and $b_l^\dagger$ and $b_l$ are the operators which create and annihilate a species $b$ with the momentum $\mathbf{p_l}$, respectively. Here we consider only the interactions conserving axion number at the leading order because the interaction rates of all of the axion number violating processes are extremely suppressed. It should be also noticed that $b$ particles are conserved as well.

Substituting Eq. (67) into Eq. (54) enables us to calculate how the occupation number of axions evolves with time due to the interactions with other particles. The $b$ particles are assumed to be in number states with a distribution $\mathcal{N}_{b,k}(t_0)$ at the initial time,

$$|\{\mathcal{N}\},\{\alpha\},\{\mathcal{N}_b\}\rangle = \prod_k \frac{1}{\sqrt{\mathcal{N}_{b,k}!V^{\mathcal{N}_{b,k}}}}(b_k^\dagger)^{\mathcal{N}_{b,k}}|\{\mathcal{N}\},\{\alpha\}\rangle, \tag{68}$$

where $|\{\mathcal{N}\},\{\alpha\}\rangle$ is given in Eq. (35). $\mathcal{N}_{b,k}$ can take a large number if $b$ is a boson, but it takes either 0 or 1 if $b$ is a fermion. In the following, the state (68) is taken in computing the expectation value in Eq. (54). Noting that

$$[H_{I,b}(t),\hat{\mathcal{N}_p}] = \frac{1}{4V^4}\sum_{jkl}\left[{\Lambda_b}^{pj}{}_{kl}e^{-i\Omega^{pj}_{kl}t}a_k^\dagger b_l^\dagger a_p b_j - \text{H.c.}\right], \tag{69}$$

we find for the condensed modes,

$$\langle[H_{I,b}(t),\hat{\mathcal{N}_p}]\rangle = \frac{1}{4V^2}\sum_j\sum_{k\le K}\left[{\Lambda_b}^{pj}{}_{kj}e^{-i(E_p-E_k)t}\mathcal{N}_{b,j}\alpha_p\alpha_k^* - \text{c.c.}\right] \quad \text{for} \quad p\le K, \tag{70}$$

which exactly vanishes because of the conservation law of three momenta $\delta_{p+j,k+j}$ in ${\Lambda_b}^{pj}{}_{kj}$. This term also vanishes for the particle-like modes,

$$\langle[H_{I,b}(t),\hat{\mathcal{N}_p}]\rangle = 0 \quad \text{for} \quad p > K. \tag{71}$$

Hence there are no contributions from interactions with other species.

From the above discussion, we conclude that the scattering does not occur, in general, between particle-like modes in the tree level of the interaction. This is inevitable consequence that follows from the two assumptions, the $b$-number conservation in Eq. (67) and the number state representation for $b$ particles in Eq. (68). Momentum transfer does not occur between number states in the tree level because of the conservation of three momenta. Schematically, this fact can be understood by using diagrams shown in Fig. 1. In the usual calculation of S-matrix for the scattering process $a+b\to a+b$, we specify "in" and "out" states as definite particle states for $a$ and $b$ species, but the momentum of each particle can differ between "in" and "out" states, as shown in Fig. 1 (a). In the in-in formalism, this tree level diagram is deformed such that "in" and "out" states are synchronized [see Fig. 1 (b)]. Then, the momenta of two "in" states must be the same if they are represented as number states. It is obvious that there is no momentum transfer in such a process, and hence it is forbidden. On the other hand, the momenta of "in" states can differ if they are coherent states, since they are not eigenstates of the number operator. Therefore, the tree level process is allowed for self-interactions between coherent states, as shown in Fig. 1 (c). This is why the second line of Eq. (62) has a non-vanishing contribution. Figure 1 (d) shows that the scattering between particle-like modes can occur in the higher order in perturbation theory. As will be seen in Appendix A, at least at the second order in perturbation theory, this process corresponds to what we calculate by using the usual Boltzmann equation.

## III. FORMATION OF AXION BOSE-EINSTEIN CONDENSATION

In this section, we discuss the cosmological evolution of dark matter axions based on the results in the previous section. The zero mode axions are produced at the time $t_1$ satisfying the condition

$$m(T_1) = 3H(t_1), \tag{72}$$

where $H(t_1)$ is the Hubble parameter at $t_1$ and $T_1$ is the temperature of radiations at that time. The temperature-dependence of axion mass is obtained in [21],

$$m^2(T) = 1.68\times10^{-7}\frac{\Lambda_{\rm QCD}^4}{F_a^2}\left(\frac{T}{\Lambda_{\rm QCD}}\right)^{-n}, \tag{73}$$

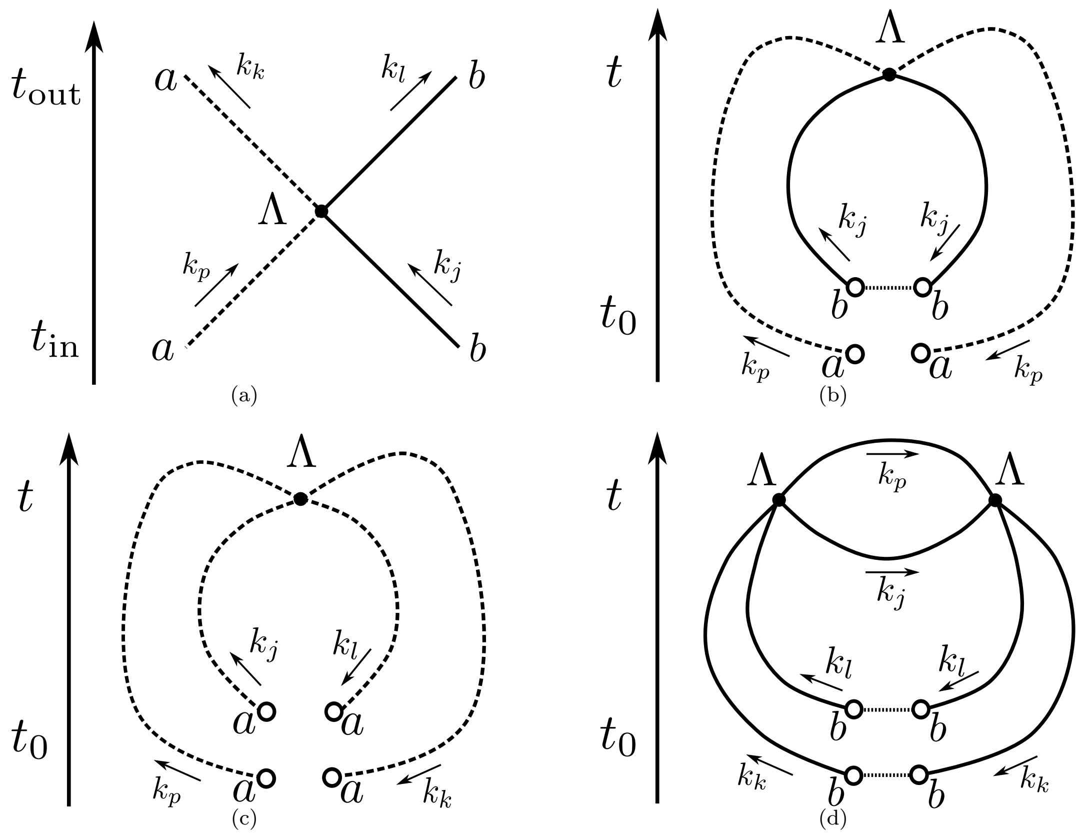


FIG. 1: Schematics of interaction processes. (a) The usual Feynmann diagram for the tree level scattering process $a+b \to a+b$. The momentum transfer occurs at the vertex denoted as $\Lambda$. (b) The tree level diagram for the process $a+b \to a+b$ in the in-in formalism. If $b$ is a particle-like mode, two momenta of "in" states at $t = t_0$ must be same. No momentum transfer occurs at the vertex. (c) Tree level diagram for the self-interaction of condensed modes $a+a \to a+a$ in the in-in formalism. Momenta of "in" states can differ from each other. Momentum transfer occurs at the vertex. (d) The diagram for the second order scattering process between particle-like modes $b+b \to b+b$ in the in-in formalism. Momenta of "in" states at $t = t_0$ are same, but momentum transfer occurs at each vertex.

with $n = 6.68$ and $\Lambda_{\rm QCD} = 400{\rm MeV}$. The time $t_1$ is estimated as [17]

$$t_1 = 3.01 \times 10^{-7}{\rm sec} \left(\frac{g_{*,1}}{70}\right)^{-n/2(4+n)} \left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{4/(4+n)} \left(\frac{\Lambda_{\rm QCD}}{400{\rm MeV}}\right)^{-2}, \tag{74}$$

where $g_{*,1}$ is the radiation degrees of freedom at the time $t_1$. This time scale $t_1$ should be identified with $t_0$, which was used in the previous section. From Eq. (50), the number density of the zero mode axions is estimated as

$$\begin{aligned} n(t) &= \frac{1}{2} m(t_1) F_a^2 X \left(\frac{R(t_1)}{R(t)}\right)^3 \\ &\simeq 2.14 \times 10^{47}{\rm cm}^{-3} X \left(\frac{g_{*,1}}{70}\right)^{n/2(4+n)} \left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{(4+2n)/(4+n)} \left(\frac{\Lambda_{\rm QCD}}{400{\rm MeV}}\right)^2 \left(\frac{R(t_1)}{R(t)}\right)^3, \end{aligned} \tag{75}$$

where $X$ is a numerical factor determined by the initial misalignment angle $(\theta^{\rm ini})^2$ and $R(t)$ is the scale factor of the universe at the time $t$. Since the momentum dispersion of axions is given by the horizon at the QCD phase transition $\delta p(t_1) \sim 1/t_1$, their velocity dispersion is estimated as

$$\delta v(t) \sim \frac{\delta p(t)}{m(0)} \sim \frac{1}{m(0) t_1} \left(\frac{R(t_1)}{R(t)}\right) \simeq 3.58 \times 10^{-4} \left(\frac{g_{*,1}}{70}\right)^{n/2(4+n)} \left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{n/(4+n)} \left(\frac{R(t_1)}{R(t)}\right), \tag{76}$$

where we have used the expression for the zero-temperature axion mass given by

$$m^2(0) = 1.46 \times 10^{-3} \frac{\Lambda_{\rm QCD}^4}{F_a^2}. \tag{77}$$

One can easily verify that the state occupation number of the zero mode axions is huge,

$$\mathcal{N} \sim n \frac{(2\pi)^3}{\frac{4\pi}{3}(m\delta v)^3} \sim 10^{61} X \left(\frac{g_{*,1}}{70}\right)^{-n/(4+n)} \left(\frac{F_a}{10^{12}\mathrm{GeV}}\right)^{2(8+n)/(4+n)} \left(\frac{\Lambda_{\rm QCD}}{400\mathrm{MeV}}\right)^{-4}. \tag{78}$$

As was assumed in Sec. II C, $K$ states around the ground state are occupied by $\mathcal{N}$ particles. Therefore, each state is occupied by $\mathcal{N}/K$ particles on average. Their thermalization rate is given by the time scale of the change of the occupation number for condensed modes [4],

$$\Gamma \equiv \frac{1}{\mathcal{N}_p(t)} \frac{d\mathcal{N}_p(t)}{dt}, \tag{79}$$

where $\mathcal{N}_p(t) \equiv \langle \hat{\mathcal{N}}_p(t) \rangle$ is given in Eq. (54) and can be estimated by using the formalism described in the previous section. Axions form a BEC if this thermalization rate exceeds the expansion rate $H(t)$ [3]. Here, it should be kept in mind whether the system is in the condensed regime or in the particle kinetic regime. For this purpose, it is necessary to compare $\Gamma$ with the typical energy dispersion of axions,

$$\delta\omega \sim \frac{1}{2} m(0)(\delta v(t))^2 \sim 3.92 \times 10^{-13}\mathrm{eV} \left(\frac{g_{*,1}}{70}\right)^{n/(4+n)} \left(\frac{F_a}{10^{12}\mathrm{GeV}}\right)^{-(4-n)/(4+n)} \left(\frac{\Lambda_{\rm QCD}}{400\mathrm{MeV}}\right)^2 \left(\frac{R(t_1)}{R(t)}\right)^2. \tag{80}$$

Let us estimate the thermalization rate in the condensed regime. Substituting the time derivative of Eq. (64) into Eq. (79) yields

$$\Gamma_{\rm condensed} \simeq \frac{1}{\mathcal{N}_p V^2} \sum_{j,k,l \le K} \mathrm{Im}\left[\Lambda_{pj}^{kl} \alpha_k \alpha_l \alpha_j^* \alpha_p^*\right], \tag{81}$$

where we have used the approximation (66). Defining the factor $\Lambda$ such that

$$\Lambda_{pj}^{kl} = \Lambda V \delta_{k+l,p+j}, \tag{82}$$

and using the approximation $\mathcal{N}_p \simeq |\alpha_p|^2 \simeq \mathcal{N}/K$, we finally obtain

$$\Gamma_{\rm condensed} \simeq \Lambda \frac{\mathcal{N}}{V} = \Lambda n(t), \tag{83}$$

where $n(t)$ is the number density of the zero mode axions given in Eq. (75). For $\lambda\phi^4$ type self-interaction, the expression of $\Lambda$ in Eq. (56) gives

$$\Gamma_{{\rm condensed},s} \simeq \frac{\lambda n(t)}{4m^2}. \tag{84}$$

On the other hand, the expression of $\Lambda$ in Eq. (58) for gravitational self-interaction leads to

$$\Gamma_{{\rm condensed},g} \simeq \frac{4\pi G m^2 n(t)}{(\delta p(t))^2}, \tag{85}$$

where $\delta p(t)$ is the momentum dispersion of axions [see Eq. (76)]. Note that these expressions are valid only if the condition $\delta\omega \ll \Gamma_{\rm condensed}$ is satisfied.

In the opposite case $\delta\omega \gg \Gamma$, the expression in the particle kinetic regime must be used. As seen in Sec. II D, the first order term vanishes in this regime, which requires us to evaluate the second order terms in order to estimate the transition rate of the occupation number. Since the thermalization rate is the quantity of $\mathcal{O}(\Lambda^2)$, it is suppressed compared with that in the condensed regime by a factor of $\Lambda$. Thus, one can expect that it is difficult for axions to thermalize in this particle kinetic regime. In fact, it is also obvious from the fact that the conditions $\Gamma_{\rm particle} > H$ (thermalization condition) and $\Gamma_{\rm particle} < \delta\omega$ (particle kinetic condition) are incompatible due to $\delta\omega < H$ after the

time $t_1$ [see Eqs. (76) and (80)]. Here, $\Gamma_{\rm particle}$ is the transition rate obtained from the second order terms in the perturbative calculation. Thus, we can conclude that the axion thermalization occurs only in the condensed regime.

Figure 2 shows the time evolution of thermalization rates $\Gamma$ together with the expansion rate $H$. We find that the transition rate $\Gamma_{{\rm condensed},g}$ due to the gravitational self-interaction exceeds the expansion rate when the temperature of photons becomes

$$T_{\rm BEC} \simeq 2.07\times 10^3 {\rm eV}\ X\left(\frac{g_{*,1}}{70}\right)^{-3n/4(4+n)}\left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{6/(4+n)}\left(\frac{\Lambda_{\rm QCD}}{400{\rm MeV}}\right), \tag{86}$$

which corresponds to the time scale

$$t_{\rm BEC} \simeq 3.09\times 10^5 {\rm sec}\ X^{-2}\left(\frac{g_{*,1}}{70}\right)^{3n/2(4+n)}\left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{-12/(4+n)}\left(\frac{\Lambda_{\rm QCD}}{400{\rm MeV}}\right)^{-2}. \tag{87}$$

Before the time $t_{\rm BEC}$, axions are decoupled from each other and are described as a classical field. Once a BEC is formed, however, axions behave like cold dark matter by different reason from that of the classical field. In particular, from the causality, the correlation length $l$ of the axion field is expected to extend over the horizon $l \lesssim t$ [4]. Hence the momentum dispersion $\delta p$ appearing in Eq. (85) becomes comparable with $l^{-1}\sim t^{-1}$, which makes the time scale of the thermalization process much faster. Then, the axion BEC continues to re-thermalize itself, and almost axions stay in the lowest energy state. This leads to the modifications of some quantities such as the energy-momentum tensor and the evolution equation of density perturbations, but they do not induce any effect on the length scale relevant to cosmological observations [3, 22].

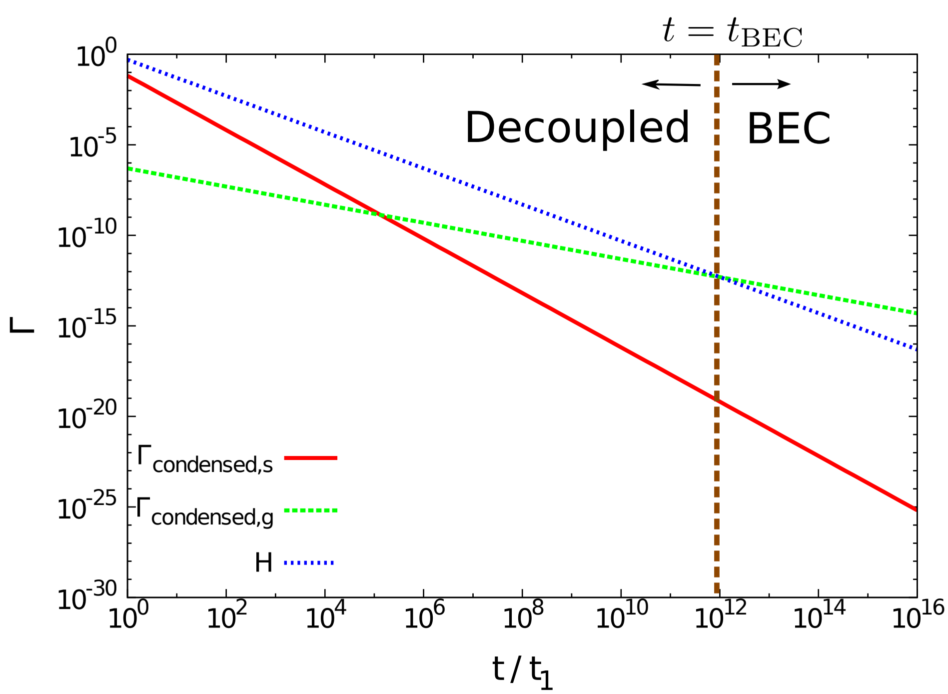


FIG. 2: The time evolution of the relaxation rates $\Gamma$ in the condensed regime and the expansion rate $H$. $\Gamma_{{\rm condensed},s}$ is the relaxation rate due to $\lambda\phi^4$ interaction estimated in Eq. (84) and $\Gamma_{{\rm condensed},g}$ is the relaxation rate due to gravitational self-interaction estimated in Eq. (85). In this figure we normalize all the scales in unit of $t_1 = 1$, where $t_1$ is given in Eq. (74), and the following parameter values $X = 1$, $g_{*,1} = 70$, and $F_a = 10^{12}$GeV are taken. Axions form a BEC at the time $t_{\rm BEC}$ given in Eq. (87).

Once the zero mode axions form a BEC, it is expected that they establish the Bose-Einstein distribution with a temperature $T_a$ different from the photon temperature $T$. This fact implies that a fraction of axions stay in the higher energy states. Then, we need to treat the thermally excited modes and zero modes separately, and the number density of axion BEC is given by $n_0 + n_T$, where $n_0$ and $n_T$ are the number density of zero modes and excited modes, respectively. The total energy density of axion BEC is also written as $mn_0 + \rho_T$ with $\rho_T$ being the energy density of thermally excited modes. Since the number density and energy density of axions just before the formation of the

BEC is given by $n(t_{\rm BEC})$ in Eq. (75) and $mn(t_{\rm BEC})$, respectively, the conservation of number and energy of axions yields $\rho_T = mn_T$, which shows that the thermally excited modes are non-relativistic. Since the momentum dispersion of axion BEC is given by the inverse of the horizon $\delta p(t_{\rm BEC}) = m\delta v(t_{\rm BEC}) \sim t_{\rm BEC}^{-1}$, the temperature of axions at $t_{\rm BEC}$ becomes

$$T_a(t_{\rm BEC}) \sim \frac{\delta p^2(t_{\rm BEC})}{3m(0)} \sim 2.48\times 10^{-37}{\rm eV}\ X^4 \left(\frac{g_{*,1}}{70}\right)^{-3n/(4+n)} \left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{(28+n)/(4+n)} \left(\frac{\Lambda_{\rm QCD}}{400{\rm MeV}}\right)^2, \tag{88}$$

which is much smaller than the mass of the axion $m \sim 10^{-6}$-$10^{-2}$eV. The number density of the thermally excited modes is estimated as

$$\begin{aligned} n_T(T_a(t_{\rm BEC})) &= \left(\frac{mT_a(t_{\rm BEC})}{2\pi}\right)^{3/2} e^{-(m-\mu)/T_a(t_{\rm BEC})} \simeq \left(\frac{mT_a(t_{\rm BEC})}{2\pi}\right)^{3/2} \\ &= 1.54\times 10^{-50}{\rm cm}^{-3}\ X^6 \left(\frac{g_{*,1}}{70}\right)^{-9n/2(4+n)} \left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{36/(4+n)} \left(\frac{\Lambda_{\rm QCD}}{400{\rm MeV}}\right)^6, \end{aligned} \tag{89}$$

where $\mu$ is the chemical potential and we used $(m-\mu)/T_a \ll 1$ for the highly degenerate case. We see that the population of thermally excited modes is negligible compared with the total number of axions,

$$\frac{n_T(T_a(t_{\rm BEC}))}{n(t_{\rm BEC})} \simeq 7.51\times 10^{-80} X^2 \left(\frac{g_{*,1}}{70}\right)^{-2n/(4+n)} \left(\frac{F_a}{10^{12}{\rm GeV}}\right)^{2(4-n)/(4+n)} \left(\frac{\Lambda_{\rm QCD}}{400{\rm MeV}}\right)^4. \tag{90}$$

Hence, almost all axions are stay in the lowest energy state.

In Refs. [4, 8], it was pointed out that these excited modes would be relativistic and affect the cosmological parameters such as the baryon to photon ratio and the effective number of neutrino species, if axions enter into thermal contact with photons. However, as shown in Sec. II E, the interaction between axions and other species exactly vanishes at the first order in perturbation theory, and hence the thermalization rate with other species is heavily suppressed, which indicates that axions forming a BEC are decoupled from other particles and do not give any significant modifications to cosmological parameters. Even though, if axion BEC is the dominant component of dark matter, they give some imprints on the structure of the inner caustics of galactic halos [7]. This might be a useful tool to distinguish axion dark matter from other particle dark matter candidates.

## IV. SUMMARY AND CONCLUSION

In this paper, we developed the formalism to describe the evolution of the zero mode axions in terms of the quantum field theoretic method. In order to solve the evolution of occupation number of axions, we used the in-in formalism and the coherent state representation for a highly degenerate Bose gas of axions. Combining these two ingredients, we derive the time evolution of the expectation value of the number operator in a perturbative way. We showed that there is non-vanishing contribution for self-interaction of condensed modes at the leading order in perturbation theory [see Eq. (64)]. On the other hand, the interactions between particle-like modes including other species exactly vanish at the first order in perturbative expansion, which indicates that their relaxation rate is suppressed.

Using the results for the time evolution of the occupation number, we estimated the thermalization rate of the zero mode axions. We recovered the expressions for thermalization rates obtained in [4], and confirmed that axions form a BEC due to the gravitational self-interactions when the temperature of photons becomes $\mathcal{O}(10^3)$eV. After the formation of axion BEC, almost all axions are in the lowest energy state and they continue to re-thermalize themselves.

From the fact that the tree level contribution vanishes for the interaction with other particle-like species, we conclude that the axion BEC has no thermal contact with other cosmological fluids, and that there is no significant effect on cosmological parameters. In particular, for the effective number of neutrino species $N_{\rm eff}$, the result of [4, 8] predicts the higher value $N_{\rm eff} = 6.77$ than the standard model, but in our analysis $N_{\rm eff}$ does not differ from the standard value. Hence the axion BEC is consistent with the standard cosmology. Only peculiar prediction is the specific phase space structure for galactic halos [7], which gives a possibility to probe axion BEC dark matter on observational grounds.

Finally, let us comment on a speculative point in our discussion. For the gravitational self-interaction of axions, we use the expression (57) which holds in the Newtonian limit. The use of this term is justified only if we are able to ignore the requirement of the causality. To be more precise, Eq. (57) can be regarded as a good approximation while the time scale of the interaction $\Gamma^{-1}$ exceeds the typical length scale $\delta l \sim \delta p^{-1}$ on which the interaction takes place. After the formation of axion BEC, however, it is expected that $\delta p \sim H$, and hence this condition becomes

$\Gamma^{-1} > H^{-1}$, which seems to be incompatible with the thermalization condition $\Gamma > H$. Therefore, in order to make a clear description about the rethermalization of axion BEC, we must extend our formalism into the expanding background, including the correction coming from general relativity. This issue is left as our future work.

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**Appendix A: Second order in perturbation theory**

In this appendix, we compute the terms of second order in $H_I$ [the third term of the right hand side of Eq. (54)]. From Eqs. (55) and (60), we obtain

$$[H_I(t),\hat{\mathcal{N}}_p] = \frac{1}{2V^4}\sum_{jkl}\left[\Lambda^{pj}_{kl}e^{-i\Omega^{pj}_{kl}t}a^\dagger_k a^\dagger_l a_j a_p - \text{H.c.}\right]. \tag{A1}$$

Then we find

$$\begin{aligned}[H_I(t_1),[H_I(t_2),\hat{\mathcal{N}}_p]] &= \frac{1}{8}\frac{1}{V^8}\sum_{mnqrjkl}\Lambda^{mn}_{qr}\Lambda^{pj}_{kl}e^{-i(\Omega^{mn}_{qr}t_1+\Omega^{pj}_{kl}t_2)}[a^\dagger_q a^\dagger_r a_m a_n, a^\dagger_k a^\dagger_l a_j a_p]\\ &\quad -\frac{1}{8}\frac{1}{V^8}\sum_{mnqrjkl}\Lambda^{mn}_{qr}\Lambda^{kl}_{pj}e^{-i(\Omega^{mn}_{qr}t_1-\Omega^{pj}_{kl}t_2)}[a^\dagger_q a^\dagger_r a_m a_n, a^\dagger_p a^\dagger_j a_k a_l].\end{aligned} \tag{A2}$$

Note that

$$\begin{aligned}[a^\dagger_q a^\dagger_r a_m a_n, a^\dagger_k a^\dagger_l a_j a_p] &= V^2\left[(\delta_{nl}\delta_{mk}+\delta_{nk}\delta_{ml})a^\dagger_q a^\dagger_r a_j a_p - (\delta_{jq}\delta_{pr}+\delta_{pq}\delta_{jr})a^\dagger_k a^\dagger_l a_m a_n\right]\\ &\quad +V\Big[\delta_{ml}a^\dagger_q a^\dagger_k a^\dagger_r a_j a_p a_n + \delta_{mk}a^\dagger_q a^\dagger_l a^\dagger_r a_j a_p a_n - \delta_{jq}a^\dagger_k a^\dagger_l a^\dagger_r a_p a_m a_n - \delta_{pq}a^\dagger_k a^\dagger_l a^\dagger_r a_j a_m a_n\\ &\quad +\delta_{nl}a^\dagger_q a^\dagger_r a^\dagger_k a_m a_j a_p + \delta_{nk}a^\dagger_q a^\dagger_r a^\dagger_l a_m a_j a_p - \delta_{jr}a^\dagger_q a^\dagger_k a^\dagger_l a_m a_p a_n - \delta_{pr}a^\dagger_q a^\dagger_k a^\dagger_l a_m a_j a_n\Big].\end{aligned} \tag{A3}$$

Using this formula, after some simplification, the expectation value of Eq. (A2) reduces to

$$\begin{aligned}\langle[H_I(t_1),[H_I(t_2),\hat{\mathcal{N}}_p]]\rangle &= \frac{1}{4}\frac{1}{V^6}\sum_{qjklm}\left[\Lambda^{qm}_{kl}\Lambda^{pj}_{qm}e^{-i(\Omega^{qm}_{kl}t_1+\Omega^{pj}_{qm}t_2)}\langle a^\dagger_k a^\dagger_l a_j a_p\rangle + \text{c.c.}\right]\\ &\quad -\frac{1}{4}\frac{1}{V^6}\sum_{qjklm}\left[\Lambda^{jq}_{pm}\Lambda^{pm}_{kl}e^{-i(\Omega^{qj}_{pm}t_1+\Omega^{pm}_{kl}t_2)}\langle a^\dagger_k a^\dagger_l a_j a_q\rangle + \text{c.c.}\right]\\ &\quad +\frac{1}{2}\frac{1}{V^7}\sum_{nmqjkl}\left[\Lambda^{ln}_{qm}\Lambda^{pj}_{kl}e^{-i(\Omega^{ln}_{qm}t_1+\Omega^{pj}_{kl}t_2)}\langle a^\dagger_q a^\dagger_m a^\dagger_k a_p a_j a_n\rangle + \text{c.c.}\right]\\ &\quad -\frac{1}{4}\frac{1}{V^7}\sum_{nmqjkl}\left[\Lambda^{mn}_{ql}\Lambda^{pl}_{kj}e^{-i(\Omega^{mn}_{ql}t_1+\Omega^{pl}_{kj}t_2)}\langle a^\dagger_k a^\dagger_j a^\dagger_q a_p a_m a_n\rangle + \text{c.c.}\right]\\ &\quad -\frac{1}{4}\frac{1}{V^7}\sum_{nmqjkl}\left[\Lambda^{mn}_{pq}\Lambda^{pj}_{kl}e^{-i(\Omega^{mn}_{pq}t_1+\Omega^{pj}_{kl}t_2)}\langle a^\dagger_k a^\dagger_l a^\dagger_q a_j a_m a_n\rangle + \text{c.c.}\right].\end{aligned} \tag{A4}$$

To compute the expectation value, we note that

$$a_{k''}a_{k'}a_k|\{\mathcal{N}\},\{\alpha\}\rangle = \begin{cases} \sqrt{\mathcal{N}_k(\mathcal{N}_k-1)(\mathcal{N}_k-2)}V^{3/2}|\{\mathcal{N}\}^{3k},\{\alpha\}\rangle & \text{if } \ k=k'=k''>K \\ \sqrt{\mathcal{N}_k\mathcal{N}_{k'}(\mathcal{N}_{k'}-1)}V^{3/2}|\{\mathcal{N}\}^{k,2k'},\{\alpha\}\rangle & \text{if } \ k\neq k',\ k'=k''>K, \text{ and } k>K \\ \sqrt{\mathcal{N}_k\mathcal{N}_{k'}\mathcal{N}_{k''}}V^{3/2}|\{\mathcal{N}\}^{k,k',k''},\{\alpha\}\rangle & \text{if } \ k\neq k'\neq k'',\ k>K,\ k'>K, \text{ and } k''>K \\ \sqrt{\mathcal{N}_k(\mathcal{N}_k-1)}\alpha_{k''}V^{3/2}|\{\mathcal{N}\}^{2k},\{\alpha\}\rangle & \text{if } \ k=k'>K \text{ and } k''\leq K \\ \sqrt{\mathcal{N}_k\mathcal{N}_{k'}}\alpha_{k''}V^{3/2}|\{\mathcal{N}\}^{kk'},\{\alpha\}\rangle & \text{if } \ k\neq k',\ k>K,\ k'>K, \text{ and } k''\leq K \\ \sqrt{\mathcal{N}_k}\alpha_{k'}\alpha_{k''}V^{3/2}|\{\mathcal{N}\}^{k},\{\alpha\}\rangle & \text{if } \ k>K,\ k'\leq K, \text{ and } k''\leq K \\ \alpha_k\alpha_{k'}\alpha_{k''}V^{3/2}|\{\mathcal{N}\},\{\alpha\}\rangle & \text{if } \ k\leq K,\ k'\leq K, \text{ and } k''\leq K \end{cases}, \tag{A5}$$

where the state $|\{\mathcal{N}\}^{3k},\{\alpha\}\rangle$ contains a factor $(a_k^\dagger)^{\mathcal{N}_k-3}/\sqrt{(\mathcal{N}_k-3)!V^{\mathcal{N}_k-3}}$ for mode $k$, the state $|\{\mathcal{N}\}^{k,2k'},\{\alpha\}\rangle$ contains a factor $(a_k^\dagger)^{\mathcal{N}_k-1}(a_{k'}^\dagger)^{\mathcal{N}_{k'}-2}/\sqrt{(\mathcal{N}_k-1)!V^{\mathcal{N}_k-1}(\mathcal{N}_{k'}-2)!V^{\mathcal{N}_{k'}-2}}$ for modes $k$ and $k'$, and the state $|\{\mathcal{N}\}^{k,k',k''},\{\alpha\}\rangle$ contains a factor $(a_k^\dagger)^{\mathcal{N}_k-1}(a_{k'}^\dagger)^{\mathcal{N}_{k'}-1}(a_{k''}^\dagger)^{\mathcal{N}_{k''}-1}/\sqrt{(\mathcal{N}_k-1)!V^{\mathcal{N}_k-1}(\mathcal{N}_{k'}-1)!V^{\mathcal{N}_{k'}-1}(\mathcal{N}_{k''}-1)!V^{\mathcal{N}_{k''}-1}}$ for modes $k$, $k'$ and $k''$. Using Eq. (A5), the expectation value of Eq. (A4) can be evaluated in a similar way to the first order terms. With a tedious but straightforward calculation, the first line of Eq. (A4) becomes

$$\begin{aligned} &\frac{1}{4}\frac{1}{V^6}\sum_{qjklm}\left[\Lambda_{kl}^{qm}\Lambda_{qm}^{pj}e^{-i(\Omega_{kl}^{qm}t_1+\Omega_{qm}^{pj}t_2)}\langle a_k^\dagger a_l^\dagger a_j a_p\rangle+\text{c.c.}\right] \\ &\quad=\frac{1}{2}\frac{1}{V^4}\sum_{qm}\sum_{k\leq K}\sum_{l>K}\left[\Lambda_{kl}^{qm}\Lambda_{qm}^{pl}e^{-i(\Omega_{kl}^{qm}t_1+\Omega_{qm}^{pl}t_2)}\alpha_k^*\alpha_p\mathcal{N}_l+\text{c.c.}\right] \\ &\quad\quad+\frac{1}{4}\frac{1}{V^4}\sum_{qm}\sum_{j,k,l\leq K}\left[\Lambda_{kl}^{qm}\Lambda_{qm}^{pj}e^{-i(\Omega_{kl}^{qm}t_1+\Omega_{qm}^{pj}t_2)}\alpha_k^*\alpha_l^*\alpha_j\alpha_p+\text{c.c.}\right]\quad\text{for}\quad p\leq K, \end{aligned} \tag{A6}$$

or

$$\begin{aligned} &\frac{1}{4}\frac{1}{V^6}\sum_{qjklm}\left[\Lambda_{kl}^{qm}\Lambda_{qm}^{pj}e^{-i(\Omega_{kl}^{qm}t_1+\Omega_{qm}^{pj}t_2)}\langle a_k^\dagger a_l^\dagger a_j a_p\rangle+\text{c.c.}\right] \\ &\quad=\frac{1}{4}\frac{1}{V^4}\sum_{qm}\left[\Lambda_{pp}^{qm}\Lambda_{qm}^{pp}e^{-i(\Omega_{pp}^{qm}t_1+\Omega_{qm}^{pp}t_2)}\mathcal{N}_p(\mathcal{N}_p-1)+\text{c.c.}\right] \\ &\quad\quad+\frac{1}{2}\frac{1}{V^4}\sum_{qm}\sum_{j>K,j\neq p}|\Lambda_{pj}^{qm}|^2\left[e^{-i\Omega_{pj}^{qm}(t_1-t_2)}\mathcal{N}_p\mathcal{N}_j+\text{c.c.}\right] \\ &\quad\quad+\frac{1}{2}\frac{1}{V^4}\sum_{qm}\sum_{j,k\leq K}\left[\Lambda_{pk}^{qm}\Lambda_{qm}^{pj}e^{-i(\Omega_{pk}^{qm}t_1+\Omega_{qm}^{pj}t_2)}\alpha_k^*\alpha_j\mathcal{N}_p+\text{c.c.}\right]\quad\text{for}\quad p>K. \end{aligned} \tag{A7}$$

The second line of Eq. (A4) becomes

$$\begin{aligned} &-\frac{1}{4}\frac{1}{V^6}\sum_{qjklm}\left[\Lambda_{pm}^{jq}\Lambda_{kl}^{pm}e^{-i(\Omega_{pm}^{qj}t_1+\Omega_{kl}^{pm}t_2)}\langle a_k^\dagger a_l^\dagger a_j a_q\rangle+\text{c.c.}\right] \\ &\quad=-\frac{1}{4}\frac{1}{V^4}\sum_{m}\sum_{q>K}\left[\Lambda_{pm}^{qq}\Lambda_{qq}^{pm}e^{-i(\Omega_{pm}^{qq}t_1+\Omega_{qq}^{pm}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)+\text{c.c.}\right] \\ &\quad\quad-\frac{1}{2}\frac{1}{V^4}\sum_{m}\sum_{j>K,j\neq p}\sum_{q>K}|\Lambda_{pm}^{jq}|^2\left[e^{-i\Omega_{pm}^{qj}(t_1-t_2)}\mathcal{N}_j\mathcal{N}_q+\text{c.c.}\right] \\ &\quad\quad-\frac{1}{V^4}\sum_{m}\sum_{j,k\leq K}\sum_{l>K}\left[\Lambda_{pm}^{jl}\Lambda_{kl}^{pm}e^{-i(\Omega_{pm}^{jl}t_1+\Omega_{kl}^{pm}t_2)}\alpha_k^*\alpha_j\mathcal{N}_l+\text{c.c.}\right] \\ &\quad\quad-\frac{1}{4}\frac{1}{V^4}\sum_{m}\sum_{j,k,l,q\leq K}\left[\Lambda_{pm}^{jq}\Lambda_{kl}^{pm}e^{-i(\Omega_{pm}^{jq}t_1+\Omega_{kl}^{pm}t_2)}\alpha_k^*\alpha_l^*\alpha_j\alpha_q+\text{c.c.}\right]. \end{aligned} \tag{A8}$$

The third line of Eq. (A4) becomes

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{V^7}\sum_{nmqjkl}\left[\Lambda^{ln}_{qm}\Lambda^{pj}_{kl}e^{-i(\Omega^{ln}_{qm}t_1+\Omega^{pj}_{kl}t_2)}\langle a^\dagger_q a^\dagger_m a^\dagger_k a_p a_j a_n\rangle+\text{c.c.}\right]\\
&=\frac{1}{V^4}\sum_l\sum_{q\le K}\sum_{k>K}\left[\Lambda^{lk}_{qk}\Lambda^{pk}_{kl}e^{-i(\Omega^{lk}_{qk}t_1+\Omega^{pk}_{kl}t_2)}\mathcal{N}_k(\mathcal{N}_k-1)\alpha^*_q\alpha_p+\text{c.c.}\right]\\
&\quad+\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\le K}\sum_{k>K}\left[\Lambda^{lk}_{kk}\Lambda^{pk}_{ql}e^{-i(\Omega^{lk}_{kk}t_1+\Omega^{pk}_{ql}t_2)}\mathcal{N}_k(\mathcal{N}_k-1)\alpha^*_q\alpha_p+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{q\le K}\sum_{m\neq k,m>K}\sum_{k>K}\Big[\mathcal{N}_m\mathcal{N}_k\alpha^*_q\alpha_p\\
&\qquad\times\left(\Lambda^{lk}_{qm}\Lambda^{pm}_{kl}e^{-i(\Omega^{lk}_{qm}t_1+\Omega^{pm}_{kl}t_2)}+\Lambda^{lm}_{qm}\Lambda^{pk}_{kl}e^{-i(\Omega^{lm}_{qm}t_1+\Omega^{pk}_{kl}t_2)}+\Lambda^{lk}_{mk}\Lambda^{pm}_{ql}e^{-i(\Omega^{lk}_{mk}t_1+\Omega^{pm}_{ql}t_2)}\right)+\text{c.c.}\Big]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{m,n,k\le K}\sum_{q>K}\left[\mathcal{N}_q\alpha^*_m\alpha^*_k\alpha_p\alpha_n\left(\Lambda^{ln}_{qm}\Lambda^{pq}_{kl}e^{-i(\Omega^{ln}_{qm}t_1+\Omega^{pq}_{kl}t_2)}+\Lambda^{lq}_{qm}\Lambda^{pn}_{kl}e^{-i(\Omega^{lq}_{qm}t_1+\Omega^{pn}_{kl}t_2)}\right)+\text{c.c.}\right]\\
&\quad+\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{m,n,k\le K}\sum_{q>K}\left[\mathcal{N}_q\alpha^*_m\alpha^*_k\alpha_p\alpha_n\left(\Lambda^{ln}_{km}\Lambda^{pq}_{ql}e^{-i(\Omega^{ln}_{km}t_1+\Omega^{pq}_{ql}t_2)}+\Lambda^{lq}_{km}\Lambda^{pn}_{ql}e^{-i(\Omega^{lq}_{km}t_1+\Omega^{pn}_{ql}t_2)}\right)+\text{c.c.}\right]\\
&\quad+\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q,m,k,j,n\le K}\left[\Lambda^{ln}_{qm}\Lambda^{pj}_{kl}e^{-i(\Omega^{ln}_{qm}t_1+\Omega^{pj}_{kl}t_2)}\alpha^*_q\alpha^*_m\alpha^*_k\alpha_p\alpha_j\alpha_n+\text{c.c.}\right]\quad\text{for}\quad p\le K,
\end{aligned}\tag{A9}
$$

or

$$
\begin{aligned}
&\frac{1}{2}\frac{1}{V^7}\sum_{nmqjkl}\left[\Lambda^{ln}_{qm}\Lambda^{pj}_{kl}e^{-i(\Omega^{ln}_{qm}t_1+\Omega^{pj}_{kl}t_2)}\langle a^\dagger_q a^\dagger_m a^\dagger_k a_p a_j a_n\rangle+\text{c.c.}\right]\\
&=\frac{1}{2}\frac{1}{V^4}\sum_l\left[\Lambda^{lp}_{pp}\Lambda^{pp}_{pl}e^{-i(\Omega^{lp}_{pp}t_1+\Omega^{pp}_{pl}t_2)}\mathcal{N}_p(\mathcal{N}_p-1)(\mathcal{N}_p-2)+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\left[\mathcal{N}_p(\mathcal{N}_p-1)\mathcal{N}_q\left(\Lambda^{lq}_{qp}\Lambda^{pp}_{pl}e^{-i(\Omega^{lq}_{qp}t_1+\Omega^{pp}_{pl}t_2)}+\Lambda^{lp}_{qp}\Lambda^{pq}_{pl}e^{-i(\Omega^{lp}_{qp}t_1+\Omega^{pq}_{pl}t_2)}\right)+\text{c.c.}\right]\\
&\quad+\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\left[\mathcal{N}_p(\mathcal{N}_p-1)\mathcal{N}_q\left(\Lambda^{lq}_{pp}\Lambda^{pp}_{ql}e^{-i(\Omega^{lq}_{pp}t_1+\Omega^{pp}_{ql}t_2)}+\Lambda^{lp}_{pp}\Lambda^{pq}_{ql}e^{-i(\Omega^{lp}_{pp}t_1+\Omega^{pq}_{ql}t_2)}\right)+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\left[\Lambda^{lq}_{pq}\Lambda^{pq}_{ql}e^{-i(\Omega^{lq}_{pq}t_1+\Omega^{pq}_{ql}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_p+\text{c.c.}\right]\\
&\quad+\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\left[\Lambda^{lq}_{qq}\Lambda^{pq}_{pl}e^{-i(\Omega^{lq}_{qq}t_1+\Omega^{pq}_{pl}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_p+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\sum_{k\neq q,k\neq p,k>K}\left[|\Lambda^{lk}_{pq}|^2e^{-i\Omega^{lk}_{pq}(t_1-t_2)}\mathcal{N}_p\mathcal{N}_q\mathcal{N}_k+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\sum_{k\neq q,k\neq p,k>K}\left[\Lambda^{lk}_{qk}\Lambda^{pq}_{pl}e^{-i(\Omega^{lk}_{qk}t_1+\Omega^{pq}_{pl}t_2)}\mathcal{N}_p\mathcal{N}_q\mathcal{N}_k+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\sum_{k\neq q,k\neq p,k>K}\left[\Lambda^{lk}_{pk}\Lambda^{pq}_{ql}e^{-i(\Omega^{lk}_{pk}t_1+\Omega^{pq}_{ql}t_2)}\mathcal{N}_p\mathcal{N}_q\mathcal{N}_k+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{n,m\le K}\left[\mathcal{N}_p(\mathcal{N}_p-1)\alpha^*_n\alpha_m\left(\Lambda^{lp}_{np}\Lambda^{pm}_{pl}e^{-i(\Omega^{lp}_{np}t_1+\Omega^{pm}_{pl}t_2)}+\Lambda^{lm}_{np}\Lambda^{pp}_{pl}e^{-i(\Omega^{lm}_{np}t_1+\Omega^{pp}_{pl}t_2)}\right)+\text{c.c.}\right]\\
&\quad+\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{n,m\le K}\left[\mathcal{N}_p(\mathcal{N}_p-1)\alpha^*_n\alpha_m\left(\Lambda^{lp}_{pp}\Lambda^{pm}_{nl}e^{-i(\Omega^{lp}_{pp}t_1+\Omega^{pm}_{nl}t_2)}+\Lambda^{lm}_{pp}\Lambda^{pp}_{nl}e^{-i(\Omega^{lm}_{pp}t_1+\Omega^{pp}_{nl}t_2)}\right)+\text{c.c.}\right]\\
&\quad+\frac{1}{V^4}\sum_l\sum_{n,m\le K}\sum_{q\neq p,q>K}\Big[\mathcal{N}_p\mathcal{N}_q\alpha^*_n\alpha_m\\
&\qquad\times\Big(\Lambda^{lq}_{np}\Lambda^{pm}_{ql}e^{-i(\Omega^{lq}_{np}t_1+\Omega^{pm}_{ql}t_2)}+\Lambda^{lq}_{nq}\Lambda^{pm}_{pl}e^{-i(\Omega^{lq}_{nq}t_1+\Omega^{pm}_{pl}t_2)}+\Lambda^{lm}_{np}\Lambda^{pq}_{ql}e^{-i(\Omega^{lm}_{np}t_1+\Omega^{pq}_{ql}t_2)}
\end{aligned}
$$

$$+\Lambda^{lm}_{nq}\Lambda^{pq}_{pl}e^{-i(\Omega^{lm}_{nq}t_1+\Omega^{pq}_{pl}t_2)}+\Lambda^{lq}_{pq}\Lambda^{pm}_{nl}e^{-i(\Omega^{lq}_{pq}t_1+\Omega^{pm}_{nl}t_2)}+\Lambda^{lm}_{pq}\Lambda^{pq}_{nl}e^{-i(\Omega^{lm}_{pq}t_1+\Omega^{pq}_{nl}t_2)}\Big)+\text{c.c.}\Big]$$
$$+\frac{1}{V^4}\sum_l\sum_{n,m,j,k\le K}\Big[\Lambda^{ln}_{pm}\Lambda^{pj}_{kl}e^{-i(\Omega^{ln}_{pm}t_1+\Omega^{pj}_{kl}t_2)}\mathcal{N}_p\alpha^*_m\alpha^*_k\alpha_j\alpha_n+\text{c.c.}\Big]$$
$$+\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{n,m,j,k\le K}\Big[\Lambda^{ln}_{km}\Lambda^{pj}_{pl}e^{-i(\Omega^{ln}_{km}t_1+\Omega^{pj}_{pl}t_2)}\mathcal{N}_p\alpha^*_m\alpha^*_k\alpha_j\alpha_n+\text{c.c.}\Big]\quad\text{for}\quad p>K. \tag{A10}$$

The fourth line of Eq. (A4) becomes

$$-\frac{1}{4}\frac{1}{V^7}\sum_{nmqjkl}\Big[\Lambda^{mn}_{ql}\Lambda^{pl}_{kj}e^{-i(\Omega^{mn}_{ql}t_1+\Omega^{pl}_{kj}t_2)}\langle a^\dagger_k a^\dagger_j a^\dagger_q a_p a_m a_n\rangle+\text{c.c.}\Big]$$
$$=-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\le K}\sum_{k>K}\Big[\Lambda^{kk}_{kl}\Lambda^{pl}_{qk}e^{-i(\Omega^{kk}_{kl}t_1+\Omega^{pl}_{qk}t_2)}\mathcal{N}_k(\mathcal{N}_k-1)\alpha^*_q\alpha_p+\text{c.c.}\Big]$$
$$-\frac{1}{4}\frac{1}{V^4}\sum_l\sum_{q\le K}\sum_{k>K}\Big[\Lambda^{kk}_{ql}\Lambda^{pl}_{kk}e^{-i(\Omega^{kk}_{ql}t_1+\Omega^{pl}_{kk}t_2)}\mathcal{N}_k(\mathcal{N}_k-1)\alpha^*_q\alpha_p+\text{c.c.}\Big]$$
$$-\frac{1}{V^4}\sum_l\sum_{q\le K}\sum_{m\neq k,m>K}\sum_{k>K}\Big[\Lambda^{km}_{ml}\Lambda^{pl}_{qk}e^{-i(\Omega^{km}_{ml}t_1+\Omega^{pl}_{qk}t_2)}\mathcal{N}_k\mathcal{N}_m\alpha^*_q\alpha_p+\text{c.c.}\Big]$$
$$-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\le K}\sum_{m\neq k,m>K}\sum_{k>K}\Big[\Lambda^{km}_{ql}\Lambda^{pl}_{km}e^{-i(\Omega^{km}_{ql}t_1+\Omega^{pl}_{km}t_2)}\mathcal{N}_k\mathcal{N}_m\alpha^*_q\alpha_p+\text{c.c.}\Big]$$
$$-\frac{1}{V^4}\sum_l\sum_{m,n,k\le K}\sum_{q>K}\Big[\Lambda^{qk}_{nl}\Lambda^{pl}_{qm}e^{-i(\Omega^{qk}_{nl}t_1+\Omega^{pl}_{qm}t_2)}\mathcal{N}_q\alpha^*_m\alpha^*_n\alpha_p\alpha_k+\text{c.c.}\Big]$$
$$-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{m,n,k\le K}\sum_{q>K}\Big[\Lambda^{qk}_{ql}\Lambda^{pl}_{mn}e^{-i(\Omega^{qk}_{ql}t_1+\Omega^{pl}_{mn}t_2)}\mathcal{N}_q\alpha^*_m\alpha^*_n\alpha_p\alpha_k+\text{c.c.}\Big]$$
$$-\frac{1}{4}\frac{1}{V^4}\sum_l\sum_{q,m,k,j,n\le K}\Big[\Lambda^{mn}_{ql}\Lambda^{pl}_{kj}e^{-i(\Omega^{mn}_{ql}t_1+\Omega^{pl}_{kj}t_2)}\alpha^*_k\alpha^*_j\alpha^*_q\alpha_p\alpha_m\alpha_n+\text{c.c.}\Big]\quad\text{for}\quad p\le K, \tag{A11}$$

or

$$-\frac{1}{4}\frac{1}{V^7}\sum_{nmqjkl}\Big[\Lambda^{mn}_{ql}\Lambda^{pl}_{kj}e^{-i(\Omega^{mn}_{ql}t_1+\Omega^{pl}_{kj}t_2)}\langle a^\dagger_k a^\dagger_j a^\dagger_q a_p a_m a_n\rangle+\text{c.c.}\Big]$$
$$=-\frac{1}{4}\frac{1}{V^4}\sum_l\Big[\Lambda^{lp}_{pp}\Lambda^{pp}_{pl}e^{-i(\Omega^{lp}_{pp}t_1+\Omega^{pp}_{pl}t_2)}\mathcal{N}_p(\mathcal{N}_p-1)(\mathcal{N}_p-2)+\text{c.c.}\Big]$$
$$-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\Big[\Lambda^{pq}_{ql}\Lambda^{pl}_{pp}e^{-i(\Omega^{pq}_{ql}t_1+\Omega^{pl}_{pp}t_2)}\mathcal{N}_p(\mathcal{N}_p-1)\mathcal{N}_q+\text{c.c.}\Big]$$
$$-\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\Big[\Lambda^{pq}_{pl}\Lambda^{pl}_{qp}e^{-i(\Omega^{pq}_{pl}t_1+\Omega^{pl}_{qp}t_2)}\mathcal{N}_p(\mathcal{N}_p-1)\mathcal{N}_q+\text{c.c.}\Big]$$
$$-\frac{1}{4}\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\Big[\Lambda^{qq}_{pl}\Lambda^{pl}_{qq}e^{-i(\Omega^{qq}_{pl}t_1+\Omega^{pl}_{qq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_p+\text{c.c.}\Big]$$
$$-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\Big[\Lambda^{qq}_{ql}\Lambda^{pl}_{pq}e^{-i(\Omega^{qq}_{ql}t_1+\Omega^{pl}_{pq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_p+\text{c.c.}\Big]$$
$$-\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\sum_{k\neq q,k\neq p,k>K}\Big[\Lambda^{kq}_{kl}\Lambda^{pl}_{pq}e^{-i(\Omega^{kq}_{kl}t_1+\Omega^{pl}_{pq}t_2)}\mathcal{N}_p\mathcal{N}_q\mathcal{N}_k+\text{c.c.}\Big]$$
$$-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{q\neq p,q>K}\sum_{k\neq q,k\neq p,k>K}\Big[|\Lambda^{qk}_{pl}|^2e^{-i\Omega^{qk}_{pl}(t_1-t_2)}\mathcal{N}_p\mathcal{N}_q\mathcal{N}_k+\text{c.c.}\Big]$$
$$-\frac{1}{V^4}\sum_l\sum_{n,m\le K}\Big[\Lambda^{mp}_{pl}\Lambda^{pl}_{np}e^{-i(\Omega^{mp}_{pl}t_1+\Omega^{pl}_{np}t_2)}\mathcal{N}_p(\mathcal{N}_p-1)\alpha^*_n\alpha_m+\text{c.c.}\Big]$$

$$
\begin{aligned}
&-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{n,m\le K}\left[\Lambda^{mp}_{nl}\Lambda^{pl}_{pp}e^{-i(\Omega^{mp}_{nl}t_1+\Omega^{pl}_{pp}t_2)}\mathcal{N}_p(\mathcal{N}_p-1)\alpha^*_n\alpha_m+\text{c.c.}\right]\\
&-\frac{1}{V^4}\sum_l\sum_{n,m\le K}\sum_{q\ne p,q>K}[\mathcal{N}_p\mathcal{N}_q\alpha^*_n\alpha_m\\
&\quad\times\left(\Lambda^{mq}_{ql}\Lambda^{pl}_{np}e^{-i(\Omega^{mq}_{ql}t_1+\Omega^{pl}_{np}t_2)}+\Lambda^{mq}_{pl}\Lambda^{pl}_{nq}e^{-i(\Omega^{mq}_{pl}t_1+\Omega^{pl}_{nq}t_2)}+\Lambda^{mq}_{nl}\Lambda^{pl}_{pq}e^{-i(\Omega^{mq}_{nl}t_1+\Omega^{pl}_{pq}t_2)}\right)+\text{c.c.}]\\
&-\frac{1}{2}\frac{1}{V^4}\sum_l\sum_{n,m,j,k\le K}\left[\Lambda^{jn}_{kl}\Lambda^{pl}_{pm}e^{-i(\Omega^{jn}_{kl}t_1+\Omega^{pl}_{pm}t_2)}\mathcal{N}_p\alpha^*_m\alpha^*_k\alpha_j\alpha_n+\text{c.c.}\right]\\
&-\frac{1}{4}\frac{1}{V^4}\sum_l\sum_{n,m,j,k\le K}\left[\Lambda^{jn}_{pl}\Lambda^{pl}_{mk}e^{-i(\Omega^{jn}_{pl}t_1+\Omega^{pl}_{mk}t_2)}\mathcal{N}_p\alpha^*_m\alpha^*_k\alpha_j\alpha_n+\text{c.c.}\right]\quad\text{for}\quad p>K.
\end{aligned}
\tag{A12}
$$

Finally, the fifth line of Eq. (A4) becomes

$$
\begin{aligned}
&-\frac{1}{4}\frac{1}{V^7}\sum_{nmqjkl}\left[\Lambda^{mn}_{pq}\Lambda^{pj}_{kl}e^{-i(\Omega^{mn}_{pq}t_1+\Omega^{pj}_{kl}t_2)}\langle a^\dagger_k a^\dagger_l a^\dagger_q a_j a_m a_n\rangle+\text{c.c.}\right]\\
&=-\frac{1}{4}\frac{1}{V^4}\sum_{q>K}\left[\Lambda^{qq}_{pq}\Lambda^{pq}_{qq}e^{-i(\Omega^{qq}_{pq}t_1+\Omega^{pq}_{qq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)(\mathcal{N}_q-2)+\text{c.c.}\right]\\
&\quad-\frac{1}{V^4}\sum_{q>K}\sum_{k\ne q,k>K}\left[\Lambda^{qk}_{qp}\Lambda^{pq}_{kq}e^{-i(\Omega^{qk}_{qp}t_1+\Omega^{pq}_{kq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_k+\text{c.c.}\right]\\
&\quad-\frac{1}{2}\frac{1}{V^4}\sum_{q>K}\sum_{k\ne q,k>K}\left[\Lambda^{qk}_{pk}\Lambda^{pq}_{qq}e^{-i(\Omega^{qk}_{pk}t_1+\Omega^{pq}_{qq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_k+\text{c.c.}\right]\\
&\quad-\frac{1}{2}\frac{1}{V^4}\sum_{q>K}\sum_{k\ne q,k>K}\left[\Lambda^{qq}_{pq}\Lambda^{pk}_{kq}e^{-i(\Omega^{qq}_{pq}t_1+\Omega^{pk}_{kq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_k+\text{c.c.}\right]\\
&\quad-\frac{1}{4}\frac{1}{V^4}\sum_{q>K}\sum_{k\ne q,k>K}\left[\Lambda^{qq}_{pk}\Lambda^{pk}_{qq}e^{-i(\Omega^{qq}_{pk}t_1+\Omega^{pk}_{qq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\mathcal{N}_k+\text{c.c.}\right]\\
&\quad-\frac{1}{V^4}\sum_{q>K}\sum_{k\ne q,k>K}\sum_{m\ne k,m\ne q,m>K}\left[\Lambda^{km}_{pm}\Lambda^{pq}_{qk}e^{-i(\Omega^{km}_{pm}t_1+\Omega^{pq}_{qk}t_2)}\mathcal{N}_q\mathcal{N}_k\mathcal{N}_m+\text{c.c.}\right]\\
&\quad-\frac{1}{2}\frac{1}{V^4}\sum_{q>K}\sum_{k\ne q,k>K}\sum_{m\ne k,m\ne q,m>K}\left[|\Lambda^{km}_{pq}|^2e^{-i\Omega^{km}_{pq}(t_1-t_2)}\mathcal{N}_q\mathcal{N}_k\mathcal{N}_m+\text{c.c.}\right]\\
&\quad-\frac{1}{V^4}\sum_{q>K}\sum_{n,m\le K}\left[\Lambda^{mq}_{pq}\Lambda^{pq}_{nq}e^{-i(\Omega^{mq}_{pq}t_1+\Omega^{pq}_{nq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\alpha^*_n\alpha_m+\text{c.c.}\right]\\
&\quad-\frac{1}{2}\frac{1}{V^4}\sum_{q>K}\sum_{n,m\le K}\left[\mathcal{N}_q(\mathcal{N}_q-1)\alpha^*_n\alpha_m\left(\Lambda^{qq}_{pq}\Lambda^{pm}_{nq}e^{-i(\Omega^{qq}_{pq}t_1+\Omega^{pm}_{nq}t_2)}+\Lambda^{mq}_{pn}\Lambda^{pq}_{qq}e^{-i(\Omega^{mq}_{pn}t_1+\Omega^{pq}_{qq}t_2)}\right)+\text{c.c.}\right]\\
&\quad-\frac{1}{4}\frac{1}{V^4}\sum_{q>K}\sum_{n,m\le K}\left[\Lambda^{qq}_{pn}\Lambda^{pm}_{qq}e^{-i(\Omega^{qq}_{pn}t_1+\Omega^{pm}_{qq}t_2)}\mathcal{N}_q(\mathcal{N}_q-1)\alpha^*_n\alpha_m+\text{c.c.}\right]\\
&\quad-\frac{1}{V^4}\sum_{q>K}\sum_{n,m\le K}\sum_{k\ne q,k>K}\left[\mathcal{N}_q\mathcal{N}_k\alpha^*_n\alpha_m\left(\Lambda^{qk}_{pk}\Lambda^{pm}_{nq}e^{-i(\Omega^{qk}_{pk}t_1+\Omega^{pm}_{nq}t_2)}\right.\right.\\
&\qquad\left.\left.+\Lambda^{mk}_{pk}\Lambda^{pq}_{nq}e^{-i(\Omega^{mk}_{pk}t_1+\Omega^{pq}_{nq}t_2)}+\Lambda^{mk}_{pq}\Lambda^{pq}_{nk}e^{-i(\Omega^{mk}_{pq}t_1+\Omega^{pq}_{nk}t_2)}+\Lambda^{mk}_{pn}\Lambda^{pq}_{qk}e^{-i(\Omega^{mk}_{pn}t_1+\Omega^{pq}_{qk}t_2)}\right)+\text{c.c.}\right]\\
&\quad-\frac{1}{2}\frac{1}{V^4}\sum_{q>K}\sum_{n,m\le K}\left[\Lambda^{qk}_{pn}\Lambda^{pm}_{qk}e^{-i(\Omega^{qk}_{pn}t_1+\Omega^{pm}_{qk}t_2)}\mathcal{N}_q\mathcal{N}_k\alpha^*_n\alpha_m+\text{c.c.}\right]\\
&\quad-\frac{1}{V^4}\sum_{q>K}\sum_{k,l,m,n\le K}\left[\Lambda^{qn}_{pl}\Lambda^{pm}_{qk}e^{-i(\Omega^{qn}_{pl}t_1+\Omega^{pm}_{qk}t_2)}\mathcal{N}_q\alpha^*_k\alpha^*_l\alpha_m\alpha_n+\text{c.c.}\right]\\
&\quad-\frac{1}{2}\frac{1}{V^4}\sum_{q>K}\sum_{k,l,m,n\le K}\left[\mathcal{N}_q\alpha^*_k\alpha^*_l\alpha_m\alpha_n\left(\Lambda^{mn}_{pl}\Lambda^{pq}_{qk}e^{-i(\Omega^{mn}_{pl}t_1+\Omega^{pq}_{qk}t_2)}+\Lambda^{qn}_{pq}\Lambda^{pm}_{kl}e^{-i(\Omega^{qn}_{pq}t_1+\Omega^{pm}_{kl}t_2)}\right)+\text{c.c.}\right]
\end{aligned}
$$

$$-\frac{1}{4}\frac{1}{V^4}\sum_{q>K}\sum_{k,l,m,n\le K}\left[\Lambda^{mn}_{pq}\Lambda^{pq}_{kl}e^{-i(\Omega^{mn}_{pq}t_1+\Omega^{pq}_{kl}t_2)}\mathcal{N}_q\alpha^*_k\alpha^*_l\alpha_m\alpha_n+\text{c.c.}\right]$$
$$-\frac{1}{4}\frac{1}{V^4}\sum_{k,l,q,j,m,n\le K}\left[\Lambda^{mn}_{pq}\Lambda^{pj}_{kl}e^{-i(\Omega^{mn}_{pq}t_1+\Omega^{pj}_{kl}t_2)}\alpha^*_k\alpha^*_l\alpha^*_q\alpha_j\alpha_m\alpha_n+\text{c.c.}\right]. \quad \text{(A13)}$$

In the particle kinetic regime, these tremendously long equations can be simplified as follows. For $p>K$, we obtain

$$\mathcal{N}_p(t) \;\simeq\; \mathcal{N}_p(t_0)-\int_{t_0}^{t}dt_2\int_{t_0}^{t_2}dt_1\langle[H_I(t_1),[H_I(t_2),\hat{\mathcal{N}}_p]]\rangle. \quad \text{(A14)}$$

In general, terms which contribute to the expectation value take a form

$$\langle[H_I(t_1),[H_I(t_2),\hat{\mathcal{N}}_p]]\rangle\propto e^{-i(\Omega_1t_1+\Omega_2t_2)}+\text{c.c.}$$

Taking the time derivative after performing the integration over $t_1$ and $t_2$, we find

$$\frac{d\mathcal{N}_p}{dt}\propto\frac{i}{\Omega_1}e^{-i(\Omega_1+\Omega_2)t}-\frac{i}{\Omega_1}e^{-i(\Omega_1t_0+\Omega_2t)}+\text{c.c.} \quad \text{(A15)}$$

If $\Omega_1+\Omega_2\neq0$, these terms drop out because of the rapidly oscillating factor in the particle kinetic regime ($\Omega^{kl}_{pq}t\to\infty$). On the other hand, if $\Omega_1+\Omega_2=0$, the first term of the right hand side of Eq. (A15) cancels with its complex conjugate. Then we obtain

$$\frac{d\mathcal{N}_p}{dt}\propto\frac{2}{\Omega_1}\sin\Omega_1(t-t_0).$$

Note that the energy conservation emerges in the limit because $\Omega_1(t-t_0)\to\infty$,

$$\frac{2}{\Omega_1}\sin\Omega_1(t-t_0)\to2\pi\delta(\Omega_1), \quad \text{(A16)}$$

which implies that terms with $\Omega_1\neq0$ do not contribute to the final result in this limit. For example, the second line of Eq. (A7) gives $\Omega_1=\Omega^{qm}_{pp}$, which does not vanish because of the conservation law of three momenta in $\Lambda^{qm}_{pp}$. The exception is the case with $q=m=p$, but the careful inspection shows that this term exactly cancels with the second line of Eq. (A8). Similar discussions are applied for the second, third, fourth, and fifth lines of Eq. (A10), the second and fourth lines of Eq. (A12), and the second, third, and sixth lines of Eq. (A13). After all, the remaining terms lead to

$$\frac{d\mathcal{N}_p}{dt}=\frac{1}{2V^4}\sum_{klq>K}|\Lambda^{kl}_{pq}|^2 2\pi\delta(\Omega^{kl}_{pq})\left[\mathcal{N}_k\mathcal{N}_l(\mathcal{N}_p+1)(\mathcal{N}_q+1)-(\mathcal{N}_k+1)(\mathcal{N}_l+1)\mathcal{N}_p\mathcal{N}_q\right], \quad \text{(A17)}$$

where we have neglected the contribution that contains the integration over condensed modes (i.e. $\sum_{q\le K}\sum_{k,l>K}$), because such terms are prohibited by the energy conservation (A16). In this way, we recover the usual Boltzmann equation [4].

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